British Journal of Mathematics & Computer Science
20(5): 1-9, 2017; Article no.BJMCS.31394
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

ms*-Modules

Ayşe Tuğba Güroğlu[1*]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Manisa Celal Bayar University, Turkey.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2017/31394
Editor(s):
(1) Dijana Mosic, Department of Mathematics, University of Nis, Serbia.
(2) Morteza Seddighin, Indiana University East Richmond, USA.
Reviewers:
(1) J. E. Macas-Daz, Universidad Autnoma de Aguascalientes, Mexico.
(2) Piyush Shroff, Texas State University, USA.
(3) Burcu Nianci Trkmen, Amasya University, Amasya.
(4) Francisco Bulnes, Tescha Research Department in Mathematics and Engineering, TESCHA, Chalco, Mexico.
Complete Peer review History: http://www.sciencedomain.org/review-history/18034

Received: 3^{rd} ***January 2017***
Accepted: 19^{th} ***February 2017***
Published: 3^{rd} ***March 2017***

Original Research Article

Abstract

Let R be a ring and M a left R-module. A module M is called ms*-module if each maximal submodule is β^* equivalent to a supplement in M. In this work, we focus on ms*-module and study various properties of this module.

Keywords: Cofinitely supplemented; cofinitely weakly supplemented; weakly supplemented; Goldie-supplemented module.*

2010 Mathematics Subject Classification: 16D10, 16D99.

1 Introduction

Throughout this paper R denotes associative ring with unity and all modules are unital left R-modules. For any module M, $Rad(M)$ and $Soc(M)$ denote the Jacobson radical and socle of M, respectively. We briefly recall some definitions and properties of (cofinitely) (weak) supplemented

Corresponding author: E-mail: tugba.guroglu@cbu.edu.tr

modules. Let M be an R-module and N, K be submodules of M. A submodule K of a module M is called *small* (or superfluous) in M, denoted by $K \ll M$, if for every submodule N of M, the equality $K + N = M$ implies $N = M$. K is called a *supplement* of N in M, if K is a minimal element in the set of submodules L of M with $N + L = M$. Equivalently, K is a supplement (weak supplement) of N in M if and only if $K + N = M$ and $K \cap N \ll K$ ($K \cap N \ll M$). A submodule K of M is called a *supplement submodule* if it is a supplement of any submodule of M. Obviously, direct summands are supplements. A module M is called *supplemented module* (*weakly supplemented*), if every submodule of M has a supplement (weak supplement) in M. A module M is $\oplus$-*supplemented module* if every submodule of M has a supplement which is a direct summand in M. A submodule N of a module M is said to be *cofinite* if $\dfrac{M}{N}$ is finitely generated. A module M is called a *cofinitely supplemented module* if every cofinite submodule of M has a supplement in M. It is clear that every supplemented module is cofinitely supplemented. In [1] Alizade and Büyükaşık defined *cofinitely weak supplemented* (or briefly cws-module) module if every cofinite submodule has a weak supplement. Obviously, cofinitely supplemented module is cofinitely weak supplemented. For additional properties and results of $\oplus$-supplemented, cofinitely supplemented and weakly supplemented modules, we refer to [1], [2], [3], [4], [5], [6].

1.1 β^* relation

In [7], G.F.Birkenmeier et.al. defined β^* relation. Some properties of β^* relation can be found in [7].

Definition 1.1. Any submodules X, Y of M are β^* equivalent, $X\beta^*Y$, if $\dfrac{X+Y}{X} \ll \dfrac{M}{X}$ and $\dfrac{X+Y}{Y} \ll \dfrac{M}{Y}$.

By ([7], Lemma 2.2), β^* is an equivalence relation and the zero submodule is β^* equivalent to any small submodule.

Proposition 1.1. *If a supplement submodule S in M is β^* equivalent to a cyclic submodule X of M, then S is also cyclic.*

Proof. Let X be a cyclic submodule and S supplement submodule of M such that $X\beta^*S$. Then there exists a submodule L of M such that $M = S + L$ and $S \cap L \ll S$. So L is weak supplement of S in M. By ([7], Theorem 2.6), L is also weak supplement of X in M. Thus we have $M = X + L$ and $X \cap L \ll M$. Since X is cyclic, we find

$$\frac{S}{S \cap L} \cong \frac{S+L}{L} = \frac{M}{L} = \frac{X+L}{L} \cong \frac{X}{X \cap L}$$

is cyclic. By the relation $S \cap L \ll S$, it follows from ([6], 11.6) that S is cyclic. □

A module M is *distributive* if for submodules A, B and C of M, $A \cap (B + C) = (A \cap B) + (A \cap C)$.

Proposition 1.2. *Let M be distributive module. Let $M = D \oplus D'$ and A, B be submodules of D. Then $A\beta^*B$ in M where B is supplement submodule in M if and only if $A\beta^*B$ in D where B is supplement submodule in D.*

Proof. ($\Rightarrow$) Suppose $A\beta^*B$ in M such that B is supplement submodule in M. Then $M = B + C$ and $B \cap C \ll B$ for some submodule C of M. Let $D = A + B + X$ for some submodule X of D. We claim that $D = A + X$ and $D = B + X$. Since $M = D \oplus D'$, then $M = A + B + X + D'$. If $A\beta^*B$ in M, then $M = A + X + D'$ and $M = B + X + D'$. By modular law, $D = D \cap M = D \cap (A + X + D') = A + X + (D \cap D') = A + X$ and $D = D \cap M = D \cap (B + X + D') = B + X$. Let us show B

supplement submodule in D. From the modularity, $D = D \cap M = D \cap (B + C) = B + (D \cap C)$ and $B \cap (D \cap C) = B \cap C \ll B$. It means that B is supplement of $D \cap C$ in D.
($\Leftarrow$) To prove the converse, suppose $A\beta^*B$ in D. Then $\dfrac{A+B}{A} \ll \dfrac{D}{A}$ and $\dfrac{A+B}{B} \ll \dfrac{D}{B}$. So $\dfrac{A+B}{A} \ll \dfrac{M}{A}$ and $\dfrac{A+B}{B} \ll \dfrac{M}{B}$ by ([6], 19.3). Thus $A\beta^*B$ in M. We shall show that B is supplement in M. Since B is supplement submodule in D, so $D = B + C$ and $B \cap C \ll B$ for some submodule C of D. Therefore $M = D \oplus D' = B + (C \oplus D')$. It follows from the distributive condition that $B \cap (C \oplus D') = (B \cap C) \oplus (B \cap D') = B \cap C \ll B$. Therefore B is supplement of $C \oplus D'$ in M. □

2 ms*-Modules

In [7], authors defined Goldie*-supplemented modules using β^* equivalence relation. A module M is *Goldie*-supplemented* (or shortly $\mathcal{G}^*$-supplemented) if for each submodule X of M, there exists a supplement submodule S of M such that $X\beta^*S$. It is clear that hollow modules and semisimple modules are $\mathcal{G}^*$-supplemented. They showed that for quasi-projective modules, supplemented and Goldie*-supplemented modules coincide.

We start this section by giving the definition of ms*-module and obtain some results of this module. We prove that if M is finitely generated refinable module and each maximal submodule is β^* equivalent to a supplement submodule, then M is $\mathcal{G}^*$-supplemented.

Definition 2.1. A module M is called *ms*-module* if, for each maximal submodule K of M, there exists a supplement submodule S of M such that $K\beta^*S$.

Example 2.1. *Local modules and semisimple modules are ms*-modules.*

Theorem 2.2. *([1], Theorem 2.16) For a module M, the following statements are equivalent.*

1. *M is cofinitely weak supplemented (cws) module,*
2. *Every maximal submodule of M has a weak supplement,*
3. *$M/cws(M)$ has no maximal submodules.*

We can use Theorem 2.2 to say ms*-modules imply cofinitely weak supplemented modules.

Proposition 2.1. *Let M be ms*-module. Then M is cofinitely weak supplemented module.*

Proof. Let K be a maximal submodule of M. It is enough to show that K has a weak supplement in M. By hypothesis, $K\beta^*S$ where S is supplement submodule in M. Then there exists a submodule L of M such that $M = S + L$ and $S \cap L \ll S$. So it follows from ([6], 19.3) that $S \cap L \ll M$. This means that L is weak supplement of S in M. By ([7], Theorem 2.6), L is also weak supplement of K in M. Hence M is cofinitely weak supplemented module. □

The following is immediate consequence of Proposition 2.1.

Corollary 2.3. *If every maximal submodule of M is β^* equivalent to a supplement which is direct summand in M, then M is cofinitely supplemented module by ([8], Theorem 2.8).*

The converse of Proposition 2.1 does not hold in general. Cofinitely weak supplemented modules need the refinable condition to become ms*-module.

A module M is said to be *refinable*, if, for any submodules U, V of M with $M = U + V$, there is a direct summand U' of M such that $U' \subseteq U$ and $M = U' + V$.

Proposition 2.2. *If M is refinable cofinitely weak supplemented, then M is ms*-module.*

Proof. We will show that for each maximal submodule of M is β^* equivalent to any supplement submodule of M. For this, let K be a maximal submodule of M. It is clear that K is cofinite submodule in M. By assumption, K has a weak supplement A in M. Then $M = K + A$ and $K \cap A \ll M$. Since M is refinable, there exists a direct summand K' of M such that $K' \subseteq K$ and $M = K' + A$. We claim that $K\beta^*K'$. If $K \cap A \ll M$, $K' \cap A \ll M$, then we have that A is weak supplement of K' in M. By ([7], Corollary 2.7), $K\beta^*K'$ where K' is direct summand supplement in M. □

Theorem 2.4. *([8], Theorem 2.8) Let R be any ring. The following statements are equivalent for an R-module M:*

1. *M is cofinitely supplemented,*
2. *Every maximal submodule of M has a supplement in M,*
3. *The module $M/Loc(M)$ does not contain a maximal submodule,*
4. *The module $M/Cof(M)$ does not contain a maximal submodule.*

Proposition 2.3. *Let M be refinable module. Then M is ms*-module if and only if M is cofinitely supplemented.*

Proof. (⇒) Let K be a maximal submodule of M. By Theorem 2.4, it is enough to show that K has a supplement in M. Since M is ms*-module, we can say that there exists a supplement submodule S of M such that $K\beta^*S$. Then $M = S + L$ and $S \cap L \ll S$ for some submodule L of M. If M is refinable, there exists a direct summand S' of M such that $S' \subseteq S$ and $M = S' + L$. So $M = S' \oplus S''$ for some submodule S'' of M. In this case, S'' is supplement of S' in M. If $S' \subseteq S$ and $S \cap L \ll S$, then $S' \cap L \ll S$. By ([6], 19.3), $S' \cap L \ll M$. This means that L is weak supplement of S' in M. If $S \cap L \ll S$, then $S \cap L \ll M$ from ([6], 19.3). By ([7], Corollary 2.7), $S\beta^*S'$. Since β^* is equivalence relation, from transitivity, if $K\beta^*S$ and $S\beta^*S'$, we get $K\beta^*S'$. It follows from ([7], Theorem 2.6) that K has a supplement S'' in M.

(⇐) Since every cofinitely supplemented module is cofinitely weak supplemented, M is ms*-module by Proposition 2.2. □

Proposition 2.4. *Let M be finitely generated ms*-module. Then M is weakly supplemented module.*

Proof. Let N be a submodule of M. Then $\dfrac{M}{N}$ is finitely generated because M is finitely generated, so N is cofinite submodule in M. By Proposition 2.1, N has a weak supplement in M. □

Corollary 2.5. *Let M be finitely generated ms*-module. Then M is semilocal.*

Proof. By ([5], Proposition 2.2) and Proposition 2.4, M is semilocal. □

Büyükaşık and Pusat Yılmaz [9] introduced the concept of ms-modules and md-modules. A module M is said to be *ms-module* if every maximal submodule of M is supplement submodule in M. A module M is called *md-module* if every maximal submodule of M is a direct summand in M. Clearly, md-module is ms*-module. Let K be a maximal submodule of M. If M is md-module, then K is direct summand in M. Since $K\beta^*K$, M is ms*-module where K is supplement submodule in M.

Proposition 2.5. *Let M be ms*-module with $Rad(M) = 0$. Then M is md-module.*

Proof. Let K be a maximal submodule of M. Now we shall prove that K is direct summand in M. Since M is ms*-module, there exists a supplement submodule S of M such that $K\beta^*S$. So we have $M = S + L$ and $S \cap L \ll S$ for some submodule L of M. Therefore $S \cap L \subseteq Rad(M)$. If $Rad(M) = 0$, then the supplement submodule S is direct summand in M, that is, $M = S \oplus L$. It is clear that L is supplement of S in M. By Theorem ([7], Theorem 2.6), L is also supplement of K in M, that is, $M = K + L$ and $K \cap L \ll L$. In a similar manner we see that $M = K \oplus L$. □

In [10], authors defined notion of maximally $\oplus$-supplemented module. A module M is called *maximally $\oplus$-supplemented* if every maximal submodule of M has a supplement that is direct summand of M. If M is ms*-module with $Rad(M) = 0$, then we can easily say that M is maximally $\oplus$-supplemented module by Proposition 2.5.

Theorem 2.6. *Let M be finitely generated refinable ms*-module. Then M is $\mathcal{G}$*-supplemented.*

Proof. Let N be a submodule of M. By Corollary 2.5, M is semilocal, that is, $\frac{M}{Rad(M)}$ is semisimple. So $\frac{M}{Rad(M)}$ is $\mathcal{G}^*$-supplemented. Then $\frac{N+Rad(M)}{Rad(M)}\beta^*\frac{X}{Rad(M)}$ such that $\frac{X}{Rad(M)}$ is supplement submodule in $\frac{M}{Rad(M)}$. Therefore by ([7], Proposition 2.9), $N\beta^*X$ in M. Let $\frac{X}{Rad(M)}$ be a supplement of $\frac{Y+Rad(M)}{Rad(M)}$ in $\frac{M}{Rad(M)}$ for some submodule Y of M. In this case, $\frac{M}{Rad(M)} = \frac{X}{Rad(M)} + \frac{Y+Rad(M)}{Rad(M)}$ and $\frac{X}{Rad(M)} \cap \frac{(Y+Rad(M))}{Rad(M)} = \frac{(X\cap Y)+Rad(M)}{Rad(M)} \ll \frac{X}{Rad(M)}$. Thus $M = X + Y$ and by ([2], 2.2), $X \cap Y \ll M$. Since M is refinable, there exists a direct summand U of M such that $U \subseteq X$ and $M = U + Y$. If $U \cap Y \subseteq X \cap Y \ll M$, then $U \cap Y \ll M$. We can say that Y is weak supplement of U in M. By ([7], Corollary 2.7), $X\beta^*U$. Since the relation β^* is equivalence relation, $N\beta^*X$ and $X\beta^*U$ imply $N\beta^*U$ such that U is supplement submodule in M. Hence M is $\mathcal{G}^*$-supplemented. □

Example 2.7. *Let M be a radical module, that is, $Rad(M) = M$. Then by ([9],* 2.3 *Proposition), M is md-module. Hence M is ms*-module.*

Example 2.8. *The $\mathbb{Z}$-module $\mathbb{Q}$ is ms*-module.*

Clearly, every $\mathcal{G}^*$-supplemented module is ms*-module. Observe that ms*-module need not to be $\mathcal{G}^*$-supplemented.

Example 2.9. *Let K be the quotient field of a discrete valuation domain R which is not complete. Let $M = K \oplus K$. Since $Rad(K) = K$, $Rad(M) = M$, that is, M is radical module. By Example 2.7, M is ms*-module but not $\mathcal{G}$*-supplemented by ([7], Example* 3.9 *(iii))*

Proposition 2.6. *M is ms*-module if and only if for maximal submodule K of M, $K = S + H$ where S is supplement submodule in M and $H \ll M$.*

Proof. ($\Rightarrow$) Let K be a maximal submodule of M. By assumption, $K\beta^*S$ where S is supplement submodule in M. Then $M = S + L$ and $S \cap L \ll S$ for some submodule L of M. So L is weak supplement of S in M. By ([7], Theorem 2.6), L is also weak supplement of K in M. In this case we have $M = K + L$ and $K \cap L \ll M$. If $M = S + L$, by modularity, $K = S + (K \cap L)$.
($\Leftarrow$) To prove M is ms*-module it is enough to show that for each maximal submodule K of M, there exists a supplement submodule S of M such that $K\beta^*S$. Let K be a maximal submodule of M. By assumption, $K = S + H$ where S is supplement submodule in M and $H \ll M$. Since β^* is equivalence relation, $K\beta^*K$, that is, $K\beta^*(S + H)$. By ([7], Corollary 2.12), we say that $K\beta^*S$. □

Proposition 2.7. *Let M be ms*-module with $Rad(M) = 0$. Then any coclosed submodule N of M with $Soc(M) \subseteq N$ is ms*-module.*

Proof. Suppose M is ms*-module with $Rad(M) = 0$. Then M is md-module by Proposition 2.5. From ([9], 2.4 Proposition), N is md-module. Therefore N is ms*-module. □

Proposition 2.8. *Let M be ms*-module. For small submodule N of M, $\frac{M}{N}$ is ms*-module.*

Proof. Let $\frac{K}{N}$ be a maximal submodule of $\frac{M}{N}$. We note that K is maximal submodule in M. By assumption, $K\beta^*S$ where S is supplement submodule in M. Let $\sigma : M \to \frac{M}{N}$ be a canonical epimorphism. By ([7], Proposition 2.9), $\frac{K}{N}\beta^*(\frac{S+N}{N})$. From ([11], Lemma 4), $\frac{S+N}{N}$ is supplement in $\frac{M}{N}$. In other words, $\frac{M}{N}$ is ms*-module. □

Corollary 2.10. *Let M be ms*-module with $Rad(M) \ll M$. Then $\frac{M}{Rad(M)}$ is ms*-module.*

For the converse of Proposition 2.8, refinable condition is needed.

Proposition 2.9. *Let $\frac{M}{N}$ be refinable ms*-module for small submodule N of M. Then M is ms*-module.*

Proof. Let K be a maximal submodule of M containing N. Then $\frac{K}{N}$ is maximal submodule in $\frac{M}{N}$. If $\frac{M}{N}$ is ms*-module, so $\frac{K}{N}\beta^*\frac{S}{N}$ where $\frac{S}{N}$ is supplement submodule in $\frac{M}{N}$. Then there exists submodule $\frac{A+N}{N}$ in $\frac{M}{N}$ such that $\frac{M}{N} = \frac{A+N}{N} + \frac{S}{N}$ and $\frac{S}{N} \cap \frac{A+N}{N} = \frac{(S\cap A)+N}{N} \ll \frac{S}{N}$. This yields $M = A + S$ and $S \cap A \ll M$, i.e., S is weak supplement of A in M. In the refinable situation, there exists a direct summand S' of M such that $S' \subseteq S$ and $M = S' + A$. Moreover, in this case, S' is a supplement submodule in M. Our claim is to show that $K\beta^*S'$. We know that $S' \cap A \subseteq S \cap A \ll M$ implies $S' \cap A \ll M$, that is, S' has a weak supplement A in M. By ([7], Corollary 2.7), $S\beta^*S'$. Let $\sigma : M \to \frac{M}{N}$ be a canonical epimorphism. By ([7], Proposition 2.9(ii)), $\sigma^{-1}(\frac{K}{N})\beta^*\sigma^{-1}(\frac{S}{N})$. Thus $K\beta^*S$. Since $S\beta^*S'$, we get $K\beta^*S'$ with the property of transitivity of β^*. □

Proposition 2.10. *Let M be projective and refinable module. Then M is ms*-module if and only if every simple factor of M has a projective cover.*

Proof. ($\Rightarrow$) Suppose that M is ms*-module. Let K be a maximal submodule of M. Then $K\beta^*S$ where S is supplement submodule in M. Then $M = S + L$ and $S \cap L \ll S$ for some submodule L of M. Since M is refinable, there exists a direct summand S' of M such that $S' \subseteq S$ and $M = S' + L$. So $S' \cap L \subseteq S \cap L \ll S$, that is, $S' \cap L \ll S$. It implies that $S' \cap L \ll M$ by ([6], 19.3). We have that L is weak supplement of S' in M. By ([7], Corollary 2.6), $S\beta^*S'$. By transitivity of β^* relation, we mention that $K\beta^*S'$. Since S' is direct summand in M, that is, $M = S' \oplus S''$ for some submodule S'' of M, by Theorem ([7], Theorem 2.6), S'' is supplement of K in M because S'' is supplement of S' in M. Thus $\frac{M}{K}$ has a projective cover from ([6], 42.1).

($\Leftarrow$) Let K be a maximal submodule of M. With the assumption we can say that $\frac{M}{K}$ has a projective cover. By ([6], 42.1), there exists a direct summand V of M such that $M = K + V$ and

$K \cap V \ll V$. Since M is refinable, there exists a direct summand K' of M such that $K' \subseteq K$ and $M = K' + V$. Then as similar to above $K' \cap V \ll M$, that is, V is weak supplement of K' in M. Since $K \cap V \ll M$, so $K\beta^*K'$ by ([7], Corollary 2.7) where K' is supplement submodule in M. □

Theorem 2.11. *Let R be a semiperfect ring. Then every projective refinable module is ms*-module.*

Proof. Let M be projective and refinable R-module. Then by ([6], 42.6), every simple factor module of M has a projective cover. Hence M is ms*-module from previous proposition. □

Proposition 2.11. *Let M be ms*-module with $Rad(M) \ll M$. Then every maximal submodule of $\frac{M}{Rad(M)}$ is direct summand. The converse holds if M is refinable.*

Proof. Let M be ms*-module with $Rad(M) \ll M$. Then by Proposition 2.1, M is cofinitely weakly supplemented. Then by ([1], Theorem 2.21), every maximal submodule of $\frac{M}{Rad(M)}$ is a direct summand. For the converse, suppose that M is refinable. Let K be a maximal submodule of M. Then $\frac{K}{Rad(M)}$ is maximal submodule of $\frac{M}{Rad(M)}$. By assumption, $\frac{M}{Rad(M)} = \frac{K}{Rad(M)} \oplus \frac{A + Rad(M)}{Rad(M)}$ for some submodule $\frac{A + Rad(M)}{Rad(M)}$ of $\frac{M}{Rad(M)}$ where A is submodule of M. So we have $M = A + K$ and $A \cap K \subseteq Rad(M)$. Since $Rad(M) \ll M$, $A \cap K \ll M$, this means that K is a weak supplement of A in M. By ([1], Theorem 2.21), M is cofinitely weakly supplemented. If M is refinable, from Proposition 2.2 we say that M is ms*-module. □

A module M is said to be *coatomic* if for every submodule N of M, $Rad(\frac{M}{N}) = \frac{M}{N}$ implies $M = N$, equivalently, every proper submodule of M is contained in a maximal submodule of M (see [12]). For instance, finitely generated and semisimple modules are coatomic. It is clear that every factor module of coatomic module is coatomic.

Proposition 2.12. *Let M be finitely generated ms*-module over a Dedekind domain and let K be a maximal submodule of M. Then the following hold:*

1. *Every weak supplement of K is coatomic.*
2. *Every maximal submodule of M is coatomic.*

Proof. 1) Let L be a weak supplement of K in M. Then $M = K + L$ and $K \cap L \ll M$. Since R is dedekind domain, by ([13], Lemma 2.1), $K \cap L$ is coatomic. Let $\sigma : \frac{M}{K} \to \frac{L}{K \cap L}$ be an isomorphism. If K is maximal submodule of M, then $\frac{M}{K}$ is simple, that is, $\frac{M}{K}$ is coatomic. Then $\frac{L}{K \cap L}$ is coatomic. Let

$$0 \to K \cap L \to L \to \frac{L}{K \cap L} \to 0$$

be an short exact sequence of modules. By ([14], Lemma 3), L is coatomic.

2) Let K be a maximal submodule of M. Since M is ms*-module, there exists a supplement submodule S in M such that $K\beta^*S$. So $\frac{K}{S} \ll \frac{M}{S}$ by definition of β^* relation. It follows from ([13], Lemma 2.1) that $\frac{K}{S}$ is coatomic. Since S is supplement submodule of M and M is finitely generated, S is finitely generated too. It implies that S is coatomic. Let

$$0 \to S \to K \to \frac{K}{S} \to 0$$

be an short exact sequence of modules. If S and $\frac{K}{S}$ are coatomic modules, then K is coatomic from ([14], Lemma 3). □

Proposition 2.13. *Let M be projective and refinable module. If M is ms*-module, then every direct summand of M is ms*-module.*

Proof. Let N be a direct summand of M. Then $M = N \oplus L$ for some submodule L of M. Since M is projective, so N is projective. By assumption and ([2], 11.31), M has the finite exchange property. Then N has the finite exchange property from ([2], 11.9). Again by ([2], 11.31), we have N is refinable. Let K be a maximal submodule of N. To use Proposition 2.10 we want to show that $\frac{N}{K}$ has a projective cover. Let $\pi : M \to N$ be a projection and $f : N \to \frac{N}{K}$ be an epimorphism. Thus $g : M \to \frac{N}{K}$ is an epimorphism such that $g = f\pi$. In this case, $Kerg = K \oplus L$. By the isomorphism theorem, $\frac{M}{K \oplus L} \cong \frac{N}{K}$ is simple. By Proposition 2.10, $\frac{M}{K \oplus L}$ has a projective cover. So $h : P \to \frac{M}{K \oplus L}$ is a projective cover of $\frac{M}{K \oplus L}$ such that $Kerh \ll P$. Let $\alpha : \frac{M}{K \oplus L} \to \frac{N}{K}$ be an isomorphism. Then $\alpha h : P \to \frac{N}{K}$ is a projective cover of $\frac{N}{K}$ with $Ker\alpha h \ll P$. By Proposition 2.10, N is ms*-module. □

Theorem 2.12. *Let M be a finitely generated refinable module. If M is projective, then the following are equivalent:*

1. *M is ms*-module,*
2. *M is cofinitely supplemented,*
3. *M is supplemented,*
4. *M is semiperfect.*

Proof. (1) $\Rightarrow$ (4) Let M be ms*-module. By Proposition 2.10, every simple factor of M has a projective cover. By ([6], 42.5), M is semiperfect.
(4) $\Rightarrow$ (3) If M is semiperfect, by ([6], 42.3), then M is supplemented since M is projective module.
(3) $\Rightarrow$ (2) It is clear that every supplemented module is also cofinitely supplemented.
(2) $\Leftrightarrow$ (1) Let M be refinable module. By Proposition 2.3, M is ms*-module if and only if M is cofinitely supplemented. □

3 Conclusion

In this work, we defined ms*-modules using β^* relation and investigate some properties of this module. From results, we see that for a refinable module, ms*-module and cofinitely supplemented modules coincide. Moreover, we obtain that every finitely generated ms*-module is semilocal and every projective refinable module is ms*-module over semiperfect ring.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Alizade R, Büyükaşık E. Cofinitely weak supplemented modules. Communication in Algebra. 2003;31(11):5377-5390.

[2] Clark J, Lomp C, Vanaja N, Wisbauer R. Lifting modules: Frontiers in mathematics. Boston, USA, Birkhäuser-Verlag; 2006.

[3] Harmancı A, Keskin D. On ⊕-supplemented modules. Acta Math. Hungarica. 1999;83:161-169.

[4] Idelhadj A, Tribak R. On some properties of ⊕-supplemented modules. Int. J. Math. Sci. 2003;69:4373-4387.

[5] Lomp C. On semilocal modules and rings. Communication in Algebra. 1999;27(4):1921-1935.

[6] Wisbauer R. Foundations of modules and rings. Gordon and Breach; 1991.

[7] Birkenmeier GF, Mutlu FT, Nebiyev C, Sökmez N, Tercan A. Goldie*-supplemented modules. Glasgow Math. Journal. 2010;52(A):41-52.

[8] Alizade R, Bilhan G, Smith P. Modules whose maximal submodules have supplements. Comm. Algebra. 2001;29(6):2389-2405.

[9] Büyükaşık E, Pusat Yılmaz D. Modules whose maximal submodules are supplement. Hacettepe Journal of Mathematics and Statistic. 2010;39(4):477-487.

[10] Türkmen E. Rings whose modules are ⊕-cofinitely supplemented. Analele Stiintifice Ale Universitatii; 2010.
DOI: 10.2478/aicu-2013-0025

[11] Acar U, Harmancı A. Principally supplemented modules. Albanian Journal of Math. 2010;4(3):79-88.

[12] Zöschinger H. Komplementierte moduln über dedekindringen. Journal of Algebra. 1974;29:42-56.

[13] Güngöroğlu G, Harmancı A. Coatomic modules over dedekind domains. Hacet. Bull. Nat. Sci. Eng. Ser. B. 1999;28:25-29.

[14] Güngöroğlu G. Coatomic modules. Far East Journal Math. Sci. 1998;2:153-162.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/18034