



A Study On Sum Formulas of Generalized Sixth-Order Linear Recurrence Sequences

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the summation formulas for generalized Hexanacci numbers are presented. As special cases, we give summation formulas of Hexanacci, Hexanacci-Lucas, sixth order Pell, sixth order Pell-Lucas, sixth order Jacobsthal, sixth order Jacobsthal-Lucas numbers.

Keywords: Hexanacci numbers; Hexanacci-Lucas numbers; sum formulas; summing formulas.

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1 INTRODUCTION

The generalized Hexanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, W_4 = c_4, W_5 = c_5, n \geq 6 \end{aligned} \tag{1.1}$$

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where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5} + \frac{1}{y}W_{-n+6}$$

for $n = 1, 2, 3, \dots$ when $y \neq 0$. Therefore, recurrence (1.1) holds for all integer n .

For some specific values of $W_0, W_1, W_2, W_3, W_4, W_5$ and r, s, t, u, v, y it is worth presenting these special Hexanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s, t, u, v, y and initial values.

Table 1. A few members of generalized Hexanacci sequences

Sequences (Numbers)	Notation	OEIS [1]
Hexanacci	$\{H_n\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1)\}$	A001592
Hexanacci-Lucas	$\{E_n\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1)\}$	A074584
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1)\}$	-
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1)\}$	-
sixth order Jacobsthal	$\{J_n^{(6)}\} = \{W_n(0, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2)\}$	-
sixth order Jacobsthal-Lucas	$\{j_n^{(6)}\} = \{W_n(2, 1, 5, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$	-

The first few values of the sequences with non-negative and negative indices are presented in the following table (Table 2).

Table 2. A few values of the sequences with positive subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
H_n	0	1	1	2	4	8	16	32	63	125	248	492	976	1936
E_n	6	1	3	7	15	31	63	120	239	475	943	1871	3711	7359
$P_n^{(6)}$	0	1	2	5	13	34	89	233	609	1592	4162	10881	28447	74371
$Q_n^{(6)}$	6	2	6	17	46	122	321	835	2182	5705	14916	38997	101953	266541
$J_n^{(6)}$	0	1	1	1	1	1	5	11	21	41	81	161	325	651
$j_n^{(6)}$	2	1	5	10	20	40	80	157	317	634	1268	2536	5072	10141

The first few values of the sequences with negative indices are presented in the following table (Table 3).

Table 3. A few values of the sequences with negative subscripts

n	1	2	3	4	5	6	7	8	9	10	11	12	13
H_{-n}	0	0	0	0	1	-1	0	0	0	0	2	-3	1
E_{-n}	-1	-1	-1	-1	-1	11	-8	-1	-1	-1	-1	23	-27
$P_{-n}^{(6)}$	0	0	0	0	1	-1	0	0	0	-1	4	-4	1
$Q_{-n}^{(6)}$	-1	-1	-1	-1	-6	17	-8	-1	-1	4	-34	65	-40
$J_{-n}^{(6)}$	$-\frac{3}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$\frac{11}{16}$	$\frac{27}{32}$	$-\frac{5}{64}$	$-\frac{197}{128}$	$-\frac{69}{256}$	$\frac{187}{512}$	$\frac{699}{1024}$	$\frac{1723}{2048}$	$-\frac{325}{4096}$	$-\frac{12613}{8192}$
$j_{-n}^{(6)}$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{23}{16}$	$\frac{25}{32}$	$\frac{25}{64}$	$\frac{25}{128}$	$\frac{25}{256}$	$\frac{25}{512}$	$-\frac{1511}{1024}$	$\frac{1561}{2048}$	$\frac{1561}{4096}$

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(6)}$.

In this work, we investigate linear summation formulas of generalized Hexanacci numbers. Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [2, 3], see also [4]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [5,6], [7,8,9,10], [11,12], [13,14] and [15], respectively.

2 LINEAR SUM FORMULAS OF GENERALIZED HEXANACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 2.1. For $n \geq 0$ we have the following formulas:

(a) (Sum of the generalized Hexanacci numbers) If $r + s + t + u + v + y - 1 \neq 0$ then

$$\sum_{k=0}^n W_k = \frac{W_{n+6} + (1-r)W_{n+5} + (1-r-s)W_{n+4} + (1-r-s-t)W_{n+3} + (1-r-s-t-u)W_{n+2} + (1-r-s-t-u-v)W_{n+1} + K_1}{r + s + t + u + v + y - 1}$$

where

$$K_1 = -W_5 + (r-1)W_4 + (r+s-1)W_3 + (r+s+t-1)W_2 + (r+s+t+u-1)W_1 + (r+s+t+u+v-1)W_0$$

(b) If $(r + s + t + u + v + y - 1)(r - s + t - u + v - y + 1) \neq 0$ then

$$\sum_{k=0}^n W_{2k} = \frac{- (s + u + y - 1) W_{2n+2} + (t + v + r(s + u + y)) W_{2n+1} + (u + y + r(t + v) - s(u + y) + (t + v)^2 - (u + y)^2) W_{2n} + (v - sv + (r + t)u + (r + t)y) W_{2n-1} + (y + (r + t)v - (s + u)y + v^2 - y^2) W_{2n-2} + y(r + t + v) W_{2n-3} + K_2}{(r + s + t + u + v + y - 1)(r - s + t - u + v - y + 1)}$$

where

$$K_2 = -(r + t + v)W_5 + (s + u + y + (r + t + v)r - 1)W_4 + ((t + v)s - (u + y)r - t - v)W_3 + (u + y + (r + t)v - (u + y)s + (r + t)^2 - (s - 1)^2)W_2 + (-v + (s + u)v - (r + t)y)W_1 + (2s + 2u + 2rv + 2tv - sy + y - uy + (r + t)^2 - (s + u)^2 + v^2 - 1)W_0$$

and

$$\sum_{k=0}^n W_{2k+1} = \frac{(r + t + v)W_{2n+2} + ((s + u + y) - (s + u + y)^2 + (t + v)^2 + r(t + v))W_{2n+1} + ((1 - s)(t + v) + r(u + y))W_{2n} + ((r + t)v + v^2 - (u + y)s + (u + y) - (u + y)^2)W_{2n-1} + ((1 - (s + u))v + (r + t)y)W_{2n-2} - y(s + u + y - 1)W_{2n-3} + K_3}{(r + s + t + u + v + y - 1)(r - s + t - u + v - y + 1)}$$

where

$$K_3 = (s + u + y - 1)W_5 - ((t + v) + (s + u + y)r)W_4 + (2s + u + y + rt + rv - su - sy + r^2 - s^2 - 1)W_3 - ((1 - s)v + (r + t)(u + y))W_2 + (2s + 2u + y + rv + tv - sy - uy + r^2 + t^2 + 2rt - s^2 - u^2 - 2su - 1)W_1 - y(r + t + v)W_0$$

(c) If $r + t + v \neq 0 \wedge s + u + y - 1 = 0$ then

$$\sum_{k=0}^n W_{2k} = \frac{W_{2n+1} + (t + v)W_{2n} + (u + y)W_{2n-1} + vW_{2n-2} + yW_{2n-3} - W_5 + rW_4 - (u + y)W_3 + (r + t)W_2 - yW_1 + (r + t + v)W_0}{r + t + v}$$

and

$$\sum_{k=0}^n W_{2k+1} = \frac{W_{2n+2} + (t + v)W_{2n+1} + (u + y)W_{2n} + vW_{2n-1} + yW_{2n-2} - W_4 + rW_3 - (u + y)W_2 + (r + t)W_1 - yW_0}{r + t + v}$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

$$yW_{n-6} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we obtain

$$\begin{aligned} yW_0 &= W_6 - rW_5 - sW_4 - tW_3 - uW_2 - vW_1 \\ yW_1 &= W_7 - rW_6 - sW_5 - tW_4 - uW_3 - vW_2 \\ yW_2 &= W_8 - rW_7 - sW_6 - tW_5 - uW_4 - vW_3 \\ yW_3 &= W_9 - rW_8 - sW_7 - tW_6 - uW_5 - vW_4 \\ &\vdots \\ yW_{n-4} &= W_{n+2} - rW_{n+1} - sW_n - tW_{n-1} - uW_{n-2} - vW_{n-3} \\ yW_{n-3} &= W_{n+3} - rW_{n+2} - sW_{n+1} - tW_n - uW_{n-1} - vW_{n-2} \\ yW_{n-2} &= W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1} - uW_n - vW_{n-1} \\ yW_{n-1} &= W_{n+5} - rW_{n+4} - sW_{n+3} - tW_{n+2} - uW_{n+1} - vW_n \\ yW_n &= W_{n+6} - rW_{n+5} - sW_{n+4} - tW_{n+3} - uW_{n+2} - vW_{n+1}. \end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned} &y \sum_{k=0}^n W_k \\ = &(W_{n+6} + W_{n+5} + W_{n+4} + W_{n+3} + W_{n+2} + W_{n+1} - W_5 - W_4 - W_3 - W_2 - W_1 - W_0 + \sum_{k=0}^n W_k) \\ &-r(W_{n+5} + W_{n+4} + W_{n+3} + W_{n+2} + W_{n+1} - W_4 - W_3 - W_2 - W_1 - W_0 + \sum_{k=0}^n W_k) \\ &-s(W_{n+4} + W_{n+3} + W_{n+2} + W_{n+1} - W_3 - W_2 - W_1 - W_0 + \sum_{k=0}^n W_k) \\ &-t(W_{n+3} + W_{n+2} + W_{n+1} - W_2 - W_1 - W_0 + \sum_{k=0}^n W_k) \\ &-u(W_{n+2} + W_{n+1} - W_1 - W_0 + \sum_{k=0}^n W_k) - v(W_{n+1} - W_0 + \sum_{k=0}^n W_k) \end{aligned}$$

and then the desired result.

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5} - yW_{n-6}$$

we obtain

$$\begin{aligned}
 rW_3 &= W_4 - sW_2 - tW_1 - uW_0 - vW_{-1} - yW_{-2} \\
 rW_5 &= W_6 - sW_4 - tW_3 - uW_2 - vW_1 - yW_0 \\
 rW_7 &= W_8 - sW_6 - tW_5 - uW_4 - vW_3 - yW_2 \\
 rW_9 &= W_{10} - sW_8 - tW_7 - uW_6 - vW_5 - yW_4 \\
 &\vdots \\
 rW_{2n-1} &= W_{2n} - sW_{2n-2} - tW_{2n-3} - uW_{2n-4} - vW_{2n-5} - yW_{2n-6} \\
 rW_{2n+1} &= W_{2n+2} - sW_{2n} - tW_{2n-1} - uW_{2n-2} - vW_{2n-3} - yW_{2n-4} \\
 rW_{2n+3} &= W_{2n+4} - sW_{2n+2} - tW_{2n+1} - uW_{2n} - vW_{2n-1} - yW_{2n-2} \\
 rW_{2n+5} &= W_{2n+6} - sW_{2n+4} - tW_{2n+3} - uW_{2n+2} - vW_{2n+1} - yW_{2n}.
 \end{aligned}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned}
 &r(-W_1 + \sum_{k=0}^n W_{2k+1}) \\
 = &(W_{2n+2} - W_2 - W_0 + \sum_{k=0}^n W_{2k}) - s(-W_0 + \sum_{k=0}^n W_{2k}) \\
 &-t(-W_{2n+1} + \sum_{k=0}^n W_{2k+1}) - u(-W_{2n} + \sum_{k=0}^n W_{2k}) \\
 &-v(-W_{2n+1} - W_{2n-1} + W_{-1} + \sum_{k=0}^n W_{2k+1}) - y(-W_{2n} - W_{2n-2} + W_{-2} + \sum_{k=0}^n W_{2k}).
 \end{aligned}$$

Since

$$\begin{aligned}
 W_{-1} &= -\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5, \\
 W_{-2} &= -\frac{v}{y}(-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5) \\
 &\quad -\frac{u}{y}W_0 - \frac{t}{y}W_1 - \frac{s}{y}W_2 - \frac{r}{y}W_3 + \frac{1}{y}W_4,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 &r(-W_1 + \sum_{k=0}^n W_{2k+1}) \tag{2.1} \\
 = &(W_{2n+2} - W_2 - W_0 + \sum_{k=0}^n W_{2k}) - s(-W_0 + \sum_{k=0}^n W_{2k}) - t(-W_{2n+1} + \sum_{k=0}^n W_{2k+1}) \\
 &-u(-W_{2n} + \sum_{k=0}^n W_{2k}) \\
 &-v(-W_{2n+1} - W_{2n-1} + (-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5) + \sum_{k=0}^n W_{2k+1}) \\
 &-y(-W_{2n} - W_{2n-2} + (-\frac{v}{y}(-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5) \\
 &\quad -\frac{u}{y}W_0 - \frac{t}{y}W_1 - \frac{s}{y}W_2 - \frac{r}{y}W_3 + \frac{1}{y}W_4) + \sum_{k=0}^n W_{2k}).
 \end{aligned}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5} - yW_{n-6}$$

we write the following obvious equations;

$$\begin{aligned} rW_2 &= W_3 - sW_1 - tW_0 - uW_{-1} - vW_{-2} - yW_{-3} \\ rW_4 &= W_5 - sW_3 - tW_2 - uW_1 - vW_0 - yW_{-1} \\ rW_6 &= W_7 - sW_5 - tW_4 - uW_3 - vW_2 - yW_1 \\ rW_8 &= W_9 - sW_7 - tW_6 - uW_5 - vW_4 - yW_3 \\ &\vdots \\ rW_{2n-2} &= W_{2n-1} - sW_{2n-3} - tW_{2n-4} - uW_{2n-5} - vW_{2n-6} - yW_{2n-7} \\ rW_{2n} &= W_{2n+1} - sW_{2n-1} - tW_{2n-2} - uW_{2n-3} - vW_{2n-4} - yW_{2n-5} \\ rW_{2n+2} &= W_{2n+3} - sW_{2n+1} - tW_{2n} - uW_{2n-1} - vW_{2n-2} - yW_{2n-3} \\ rW_{2n+4} &= W_{2n+5} - sW_{2n+3} - tW_{2n+2} - uW_{2n+1} - vW_{2n} - yW_{2n-1} \\ rW_{2n+6} &= W_{2n+7} - sW_{2n+5} - tW_{2n+4} - uW_{2n+3} - vW_{2n+2} - yW_{2n+1}. \end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$\begin{aligned} r(-W_0 + \sum_{k=0}^n W_{2k}) &= (-W_1 + \sum_{k=0}^n W_{2k+1}) - s(-W_{2n+1} + \sum_{k=0}^n W_{2k+1}) - t(-W_{2n} + \sum_{k=0}^n W_{2k}) \\ &\quad - u(-W_{2n+1} - W_{2n-1} + W_{-1} + \sum_{k=0}^n W_{2k+1}) - v(-W_{2n} - W_{2n-2} + W_{-2} + \sum_{k=0}^n W_{2k}) \\ &\quad - y(-W_{2n+1} - W_{2n-1} - W_{2n-3} + W_{-3} + W_{-1} + \sum_{k=0}^n W_{2k+1}). \end{aligned} \tag{2.2}$$

Using

$$\begin{aligned} W_{-1} &= (-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5), \\ W_{-2} &= (-\frac{v}{y}(-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5) - \frac{u}{y}W_0 - \frac{t}{y}W_1 - \frac{s}{y}W_2 - \frac{r}{y}W_3 + \frac{1}{y}W_4), \\ W_{-3} &= (-\frac{v}{y}(-\frac{v}{y}(-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5) - \frac{u}{y}W_0 - \frac{t}{y}W_1 - \frac{s}{y}W_2 - \frac{r}{y}W_3 + \frac{1}{y}W_4) \\ &\quad - \frac{u}{y}(-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5) - \frac{t}{y}W_0 - \frac{s}{y}W_1 - \frac{r}{y}W_2 + \frac{1}{y}W_3), \end{aligned}$$

and solving the system (2.1)-(2.2), the required result of (b) and (c) follow.

Taking $r = s = t = u = v = y = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.1. *If $r = s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:*

- (a) $\sum_{k=0}^n W_k = \frac{1}{5}(W_{n+6} - W_{n+4} - 2W_{n+3} - 3W_{n+2} - 4W_{n+1} - W_5 + W_3 + 2W_2 + 3W_1 + 4W_0).$
- (b) $\sum_{k=0}^n W_{2k} = \frac{1}{5}(-2W_{2n+2} + 5W_{2n+1} + 2W_{2n} + 4W_{2n-1} + W_{2n-2} + 3W_{2n-3} - 3W_5 + 5W_4 - 2W_3 + 6W_2 - W_1 + 7W_0).$
- (c) $\sum_{k=0}^n W_{2k+1} = \frac{1}{5}(3W_{2n+2} + 2W_{2n} - W_{2n-1} + W_{2n-2} - 2W_{2n-3} + 2W_5 - 5W_4 + 3W_3 - 4W_2 + 4W_1 - 3W_0).$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 2.2. For $n \geq 0$, Hexanacci numbers have the following properties:

- (a) $\sum_{k=0}^n H_k = \frac{1}{5}(H_{n+6} - H_{n+4} - 2H_{n+3} - 3H_{n+2} - 4H_{n+1} - 1)$.
- (b) $\sum_{k=0}^n H_{2k} = \frac{1}{5}(-2H_{2n+2} + 5H_{2n+1} + 2H_{2n} + 4H_{2n-1} + H_{2n-2} + 3H_{2n-3} - 3)$.
- (c) $\sum_{k=0}^n H_{2k+1} = \frac{1}{5}(3H_{2n+2} + 2H_{2n} - H_{2n-1} + H_{2n-2} - 2H_{2n-3} + 2)$

Taking $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Hexanacci-Lucas numbers.

Corollary 2.3. For $n \geq 0$, Hexanacci-Lucas numbers have the following properties:

- (a) $\sum_{k=0}^n E_k = \frac{1}{5}(E_{n+6} - E_{n+4} - 2E_{n+3} - 3E_{n+2} - 4E_{n+1} + 9)$.
- (b) $\sum_{k=0}^n E_{2k} = \frac{1}{5}(-2E_{2n+2} + 5E_{2n+1} + 2E_{2n} + 4E_{2n-1} + E_{2n-2} + 3E_{2n-3} + 27)$.
- (c) $\sum_{k=0}^n E_{2k+1} = \frac{1}{5}(3E_{2n+2} + 2E_{2n} - E_{2n-1} + E_{2n-2} - 2E_{2n-3} - 18)$.

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.2. If $r = 2, s = t = u = v = y = 1$ then for $n \geq 0$ we have the following formulas:

- (a) $\sum_{k=0}^n W_k = \frac{1}{6}(W_{n+6} - W_{n+5} - 2W_{n+4} - 3W_{n+3} - 4W_{n+2} - 5W_{n+1} - W_5 + W_4 + 2W_3 + 3W_2 + 4W_1 + 5W_0)$.
- (b) $\sum_{k=0}^n W_{2k} = \frac{1}{6}(-W_{2n+2} + 4W_{2n+1} + 2W_{2n} + 3W_{2n-1} + W_{2n-2} + 2W_{2n-3} - 2W_5 + 5W_4 - 2W_3 + 6W_2 - W_1 + 7W_0)$.
- (c) $\sum_{k=0}^n W_{2k+1} = \frac{1}{6}(2W_{2n+2} + W_{2n+1} + 2W_{2n} + W_{2n-2} - W_{2n-3} + W_5 - 4W_4 + 4W_3 - 3W_2 + 5W_1 - 2W_0)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of sixth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$).

Corollary 2.4. For $n \geq 0$, sixth-order Pell numbers have the following properties:

- (a) $\sum_{k=0}^n P_k = \frac{1}{6}(P_{n+6} - P_{n+5} - 2P_{n+4} - 3P_{n+3} - 4P_{n+2} - 5P_{n+1} - 1)$.
- (b) $\sum_{k=0}^n P_{2k} = \frac{1}{6}(-P_{2n+2} + 4P_{2n+1} + 2P_{2n} + 3P_{2n-1} + P_{2n-2} + 2P_{2n-3} - 2)$.
- (c) $\sum_{k=0}^n P_{2k+1} = \frac{1}{6}(2P_{2n+2} + P_{2n+1} + 2P_{2n} + P_{2n-2} - P_{2n-3} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$ in the last Proposition, we have the following Corollary which presents linear sum formulas of sixth-order Pell-Lucas numbers.

Corollary 2.5. For $n \geq 0$, sixth-order Pell-Lucas numbers have the following properties:

- (a) $\sum_{k=0}^n Q_k = \frac{1}{6}(Q_{n+6} - Q_{n+5} - 2Q_{n+4} - 3Q_{n+3} - 4Q_{n+2} - 5Q_{n+1} + 14)$.
- (b) $\sum_{k=0}^n Q_{2k} = \frac{1}{6}(-Q_{2n+2} + 4Q_{2n+1} + 2Q_{2n} + 3Q_{2n-1} + Q_{2n-2} + 2Q_{2n-3} + 28)$.
- (c) $\sum_{k=0}^n Q_{2k+1} = \frac{1}{6}(2Q_{2n+2} + Q_{2n+1} + 2Q_{2n} + Q_{2n-2} - Q_{2n-3} - 14)$

If $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ then $(r + s + t + u + v + y - 1)(r - s + t - u + v - y + 1) = 0$ so we can't use Theorem 2.1 (b).

Proposition 2.3. If $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ then for $n \geq 0$ we have the following formula:

$$\sum_{k=0}^n W_k = \frac{1}{6}(W_{n+6} - W_{n+4} - 2W_{n+3} - 3W_{n+2} - 4W_{n+1} - W_5 + W_3 + 2W_2 + 3W_1 + 4W_0).$$

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Proposition, we have the following Corollary which presents linear sum formula of sixth-order Jacobsthal numbers.

Corollary 2.6. For $n \geq 0$, sixth order Jacobsthal numbers have the following property:

$$\sum_{k=0}^n J_k = \frac{1}{6}(J_{n+6} - J_{n+4} - 2J_{n+3} - 3J_{n+2} - 4J_{n+1} + 5).$$

From the last Proposition, we have the following Corollary which gives linear sum formula of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

Corollary 2.7. For $n \geq 0$, sixth order Jacobsthal-Lucas numbers have the following property:

$$\sum_{k=0}^n j_k = \frac{1}{6}(j_{n+6} - j_{n+4} - 2j_{n+3} - 3j_{n+2} - 4j_{n+1} - 9).$$

3 LINEAR SUM FORMULAS OF GENERALIZED HEXANACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 3.1. For $n \geq 1$ we have the following formulas:

(a) (Sum of the generalized Hexanacci numbers with negative indices) If $r + s + t + u + v + y - 1 \neq 0$, then

$$\sum_{k=1}^n W_{-k} = \frac{-W_{-n+5} + (r-1)W_{-n+4} + (r+s-1)W_{-n+3} + (r+s+t-1)W_{-n+2} + (r+s+t+u-1)W_{-n+1} + (r+s+t+u+v-1)W_{-n} + K_4}{r+s+t+u+v+y-1}$$

where

$$K_4 = W_5 + (1-r)W_4 + (1-r-s)W_3 + (1-r-s-t)W_2 + (1-r-s-t-u)W_1 + (1-r-s-t-u-v)W_0$$

(b) If $(r-s+t-u+v-y+1)(r+s+t+u+v+y-1) \neq 0$ then

$$\sum_{k=1}^n W_{-2k} = \frac{(s+u+y-1)W_{-2n+4} - (t+v+(s+u+y)r)W_{-2n+3} + (u+y-(u+y)s+(t+v+r)r-(s-1)^2)W_{-2n+2} - ((r+t)u+(r+t)y+(1-s)v)W_{-2n+1} + (y+2(s+u)+(r+t)v-(s+u)y+(r+t)^2-(s+u)^2-1)W_{-2n} - y(r+t+v)W_{-2n-1} + K_5}{(r-s+t-u+v-y+1)(r+s+t+u+v+y-1)}$$

where

$$K_5 = (r+t+v)W_5 - (s+u+y+(t+v+r)r-1)W_4 + ((u+y)r+(1-s)(t+v))W_3 + (-u-y-(r+t)v+(u+y)s-(r+t)^2+(s-1)^2)W_2 + (v+(r+t)y-(s+u)v)W_1 + (-y-2(s+u)-2(r+t)v+(s+u)y-(r+t)^2+(s+u)^2-v^2+1)W_0$$

and

$$\sum_{k=1}^n W_{-2k+1} = \frac{-(r+t+v)W_{-2n+4} + (s+u+y+(t+v+r)r-1)W_{-2n+3} + ((s-1)(t+v)-(u+y)r)W_{-2n+2} + (u+y+(r+t)v-(u+y)s+(r+t)^2-(s-1)^2)W_{-2n+1} + (-v-(r+t)y+(s+u)v)W_{-2n} + y(s+u+y-1)W_{-2n-1} + K_6}{(r-s+t-u+v-y+1)(r+s+t+u+v+y-1)}$$

where

$$K_6 = -(s+u+y-1)W_5 + (t+v+(s+u+y)r)W_4 + (-u-y-(r+t+v)r+(u+y)s+(s-1)^2)W_3 + ((u+y)r+(u+y)t+(1-s)v)W_2 + (-y-2(s+u)-(r+t)v+(s+u)y-(r+t)^2+(s+u)^2+1)W_1 + y(r+t+v)W_0$$

(c) If $r + t + v \neq 0 \wedge s + u + y - 1 = 0$ then

$$\sum_{k=1}^n W_{-2k} = \frac{-W_{-2n+3} + rW_{-2n+2} - (u+y)W_{-2n+1} - yW_{-2n-1} + (r+t)W_{-2n} + W_5 - rW_4 + (u+y)W_3 - (r+t)W_2 + yW_1 - (r+t+v)W_0}{r+t+v}$$

and

$$\sum_{k=1}^n W_{-2k+1} = \frac{-W_{-2n+4} + rW_{-2n+3} - (u+y)W_{-2n+2} + (r+t)W_{-2n+1} - yW_{-2n} + W_4 - rW_3 + (u+y)W_2 - (t+r)W_1 + yW_0}{r+t+v}$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_{-n} = \frac{1}{y}W_{-n+6} - \frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5}$$

i.e.

$$yW_{-n} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - vW_{-n+1}$$

we obtain

$$\begin{aligned} yW_{-n} &= W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - vW_{-n+1} \\ yW_{-n+1} &= W_{-n+7} - rW_{-n+6} - sW_{-n+5} - tW_{-n+4} - uW_{-n+3} - vW_{-n+2} \\ yW_{-n+2} &= W_{-n+8} - rW_{-n+7} - sW_{-n+6} - tW_{-n+5} - uW_{-n+4} - vW_{-n+3} \\ &\vdots \\ yW_{-4} &= W_2 - rW_1 - sW_0 - tW_{-1} - uW_{-2} - vW_{-3} \\ yW_{-3} &= W_3 - rW_2 - sW_1 - tW_0 - uW_{-1} - vW_{-2} \\ yW_{-2} &= W_4 - rW_3 - sW_2 - tW_1 - uW_0 - vW_{-1} \\ yW_{-1} &= W_5 - rW_4 - sW_3 - tW_2 - uW_1 - vW_0. \end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned} &y\left(\sum_{k=1}^n W_{-k}\right) \\ = &(-W_{-n+5} - W_{-n+4} - W_{-n+3} - W_{-n+2} - W_{-n+1} - W_{-n} + W_5 + W_4 + W_3 + W_2 + W_1 + W_0 + \sum_{k=1}^n W_{-k}) \\ &-r(-W_{-n+4} - W_{-n+3} - W_{-n+2} - W_{-n+1} - W_{-n} + W_4 + W_3 + W_2 + W_1 + W_0 + \sum_{k=1}^n W_{-k}) \\ &-s(-W_{-n+3} - W_{-n+2} - W_{-n+1} - W_{-n} + W_3 + W_2 + W_1 + W_0 + \sum_{k=1}^n W_{-k}) \\ &-t(-W_{-n+2} - W_{-n+1} - W_{-n} + W_2 + W_1 + W_0 + \sum_{k=1}^n W_{-k}) \\ &-u(-W_{-n+1} - W_{-n} + W_1 + W_0 + \sum_{k=1}^n W_{-k}) - v(-W_{-n} + W_0 + \sum_{k=1}^n W_{-k}) \end{aligned}$$

and then the desired result.

(b) and (c) Using the recurrence relation

$$W_{-n} = \frac{1}{y}W_{-n+6} - \frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5}$$

i.e.

$$vW_{-n+1} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - yW_{-n}$$

we obtain

$$\begin{aligned} vW_{-2n+1} &= W_{-2n+6} - rW_{-2n+5} - sW_{-2n+4} - tW_{-2n+3} - uW_{-2n+2} - yW_{-2n} \\ vW_{-2n+3} &= W_{-2n+8} - rW_{-2n+7} - sW_{-2n+6} - tW_{-2n+5} - uW_{-2n+4} - yW_{-2n+2} \\ vW_{-2n+5} &= W_{-2n+10} - rW_{-2n+9} - sW_{-2n+8} - tW_{-2n+7} - uW_{-2n+6} - yW_{-2n+4} \\ vW_{-2n+7} &= W_{-2n+12} - rW_{-2n+11} - sW_{-2n+10} - tW_{-2n+9} - uW_{-2n+8} - yW_{-2n+6} \\ &\vdots \\ vW_{-7} &= W_{-2} - rW_{-3} - sW_{-4} - tW_{-5} - uW_{-6} - yW_{-8} \\ vW_{-5} &= W_0 - rW_{-1} - sW_{-2} - tW_{-3} - uW_{-4} - yW_{-6} \\ vW_{-3} &= W_2 - rW_1 - sW_0 - tW_{-1} - uW_{-2} - yW_{-4} \\ vW_{-1} &= W_4 - rW_3 - sW_2 - tW_1 - uW_0 - yW_{-2}. \end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned} &v \sum_{k=1}^n W_{-2k+1} \tag{3.1} \\ &= (-W_{-2n+4} - W_{-2n+2} - W_{-2n} + W_0 + W_2 + W_4 + \sum_{k=1}^n W_{-2k}) \\ &\quad - r(-W_{-2n+3} - W_{-2n+1} + W_3 + W_1 + \sum_{k=1}^n W_{-2k+1}) - s(-W_{-2n+2} - W_{-2n} + W_0 + W_2 + \sum_{k=1}^n W_{-2k}) \\ &\quad - t(-W_{-2n+1} + W_1 + \sum_{k=1}^n W_{-2k+1}) - u(-W_{-2n} + W_0 + \sum_{k=1}^n W_{-2k}) - y(\sum_{k=1}^n W_{-2k}). \end{aligned}$$

Similarly, using the recurrence relation

$$W_{-n} = \frac{1}{y}W_{-n+6} - \frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5}$$

i.e.

$$vW_{-n+1} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - yW_{-n}$$

we obtain

$$\begin{aligned} vW_{-2n} &= W_{-2n+5} - rW_{-2n+4} - sW_{-2n+3} - tW_{-2n+2} - uW_{-2n+1} - yW_{-2n-1} \\ vW_{-2n+2} &= W_{-2n+7} - rW_{-2n+6} - sW_{-2n+5} - tW_{-2n+4} - uW_{-2n+3} - yW_{-2n+1} \\ vW_{-2n+4} &= W_{-2n+9} - rW_{-2n+8} - sW_{-2n+7} - tW_{-2n+6} - uW_{-2n+5} - yW_{-2n+3} \\ vW_{-2n+6} &= W_{-2n+11} - rW_{-2n+10} - sW_{-2n+9} - tW_{-2n+8} - uW_{-2n+7} - yW_{-2n+5} \\ &\vdots \\ vW_{-8} &= W_{-3} - rW_{-4} - sW_{-5} - tW_{-6} - uW_{-7} - yW_{-9} \\ vW_{-6} &= W_{-1} - rW_{-2} - sW_{-3} - tW_{-4} - uW_{-5} - yW_{-7} \\ vW_{-4} &= W_1 - rW_0 - sW_{-1} - tW_{-2} - uW_{-3} - yW_{-5} \\ vW_{-2} &= W_3 - rW_2 - sW_1 - tW_0 - uW_{-1} - yW_{-3}. \end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
 v \sum_{k=1}^n W_{-2k} &= (-W_{-2n+3} - W_{-2n+1} + W_1 + W_3 + \sum_{k=1}^n W_{-2k+1}) \\
 &\quad -r(-W_{-2n+2} - W_{-2n} + W_2 + W_0 + \sum_{k=1}^n W_{-2k}) \\
 &\quad -s(-W_{-2n+1} + W_1 + \sum_{k=1}^n W_{-2k+1}) - t(-W_{-2n} + W_0 + \sum_{k=1}^n W_{-2k}) \\
 &\quad -u(\sum_{k=1}^n W_{-2k+1}) - y(W_{-2n-1} - W_{-1} + \sum_{k=1}^n W_{-2k+1}).
 \end{aligned} \tag{3.2}$$

Then, using

$$W_{-1} = \left(-\frac{v}{y}W_0 - \frac{u}{y}W_1 - \frac{t}{y}W_2 - \frac{s}{y}W_3 - \frac{r}{y}W_4 + \frac{1}{y}W_5\right)$$

and solving system (3.1)-(3.2) the required result of (b) and (c) follow.

Taking $r = s = t = u = v = y = 1$ in Theorem 3.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 3.1. *If $r = s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

- (a) $\sum_{k=1}^n W_{-k} = \frac{1}{5}(-W_{-n+5} + W_{-n+3} + 2W_{-n+2} + 3W_{-n+1} + 4W_{-n} + W_5 - W_3 - 2W_2 - 3W_1 - 4W_0)$.
- (b) $\sum_{k=1}^n W_{-2k} = \frac{1}{5}(2W_{-2n+4} - 5W_{-2n+3} + 3W_{-2n+2} - 4W_{-2n+1} + 4W_{-2n} - 3W_{-2n-1} + 3W_5 - 5W_4 + 2W_3 - 6W_2 + W_1 - 7W_0)$.
- (c) $\sum_{k=1}^n W_{-2k+1} = \frac{1}{5}(-3W_{-2n+4} + 5W_{-2n+3} - 2W_{-2n+2} + 6W_{-2n+1} - W_{-2n} + 2W_{-2n-1} - 2W_5 + 5W_4 - 3W_3 + 4W_2 - 4W_1 + 3W_0)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Hexanacci numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$).

Corollary 3.2. *For $n \geq 1$, Hexanacci numbers have the following properties:*

- (a) $\sum_{k=1}^n H_{-k} = \frac{1}{5}(-H_{-n+5} + H_{-n+3} + 2H_{-n+2} + 3H_{-n+1} + 4H_{-n} + 1)$.
- (b) $\sum_{k=1}^n H_{-2k} = \frac{1}{5}(2H_{-2n+4} - 5H_{-2n+3} + 3H_{-2n+2} - 4H_{-2n+1} + 4H_{-2n} - 3H_{-2n-1} + 3)$.
- (c) $\sum_{k=1}^n H_{-2k+1} = \frac{1}{5}(-3H_{-2n+4} + 5H_{-2n+3} - 2H_{-2n+2} + 6H_{-2n+1} - H_{-2n} + 2H_{-2n-1} - 2)$.

Taking $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Hexanacci-Lucas numbers.

Corollary 3.3. *For $n \geq 1$, Hexanacci-Lucas numbers have the following properties:*

- (a) $\sum_{k=1}^n E_{-k} = \frac{1}{5}(-E_{-n+5} + E_{-n+3} + 2E_{-n+2} + 3E_{-n+1} + 4E_{-n} - 9)$.
- (b) $\sum_{k=1}^n E_{-2k} = \frac{1}{5}(2E_{-2n+4} - 5E_{-2n+3} + 3E_{-2n+2} - 4E_{-2n+1} + 4E_{-2n} - 3E_{-2n-1} - 27)$.
- (c) $\sum_{k=1}^n E_{-2k+1} = \frac{1}{5}(-3E_{-2n+4} + 5E_{-2n+3} - 2E_{-2n+2} + 6E_{-2n+1} - E_{-2n} + 2E_{-2n-1} + 18)$.

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 3.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 3.2. *If $r = 2, s = t = u = v = y = 1$ then for $n \geq 1$ we have the following formulas:*

- (a) $\sum_{k=1}^n W_{-k} = \frac{1}{6}(-W_{-n+5} + W_{-n+4} + 2W_{-n+3} + 3W_{-n+2} + 4W_{-n+1} + 5W_{-n} + W_5 - W_4 - 2W_3 - 3W_2 - 4W_1 - 5W_0)$.
- (b) $\sum_{k=1}^n W_{-2k} = \frac{1}{6}(W_{-2n+4} - 4W_{-2n+3} + 4W_{-2n+2} - 3W_{-2n+1} + 5W_{-2n} - 2W_{-2n-1} + 2W_5 - 5W_4 + 2W_3 - 6W_2 + W_1 - 7W_0)$.
- (c) $\sum_{k=1}^n W_{-2k+1} = \frac{1}{6}(-2W_{-2n+4} + 5W_{-2n+3} - 2W_{-2n+2} + 6W_{-2n+1} - W_{-2n} + W_{-2n-1} - W_5 + 4W_4 - 4W_3 + 3W_2 - 5W_1 + 2W_0)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of sixth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$).

Corollary 3.4. For $n \geq 1$, sixth-order Pell numbers have the following properties:

- (a) $\sum_{k=1}^n P_{-k} = \frac{1}{6}(-P_{-n+5} + P_{-n+4} + 2P_{-n+3} + 3P_{-n+2} + 4P_{-n+1} + 5P_{-n} + 1)$.
- (b) $\sum_{k=1}^n P_{-2k} = \frac{1}{6}(P_{-2n+4} - 4P_{-2n+3} + 4P_{-2n+2} - 3P_{-2n+1} + 5P_{-2n} - 2P_{-2n-1} + 2)$.
- (c) $\sum_{k=1}^n P_{-2k+1} = \frac{1}{6}(-2P_{-2n+4} + 5P_{-2n+3} - 2P_{-2n+2} + 6P_{-2n+1} - P_{-2n} + P_{-2n-1} - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$ in the last Proposition, we have the following Corollary which presents linear sum formulas of sixth-order Pell-Lucas numbers.

Corollary 3.5. For $n \geq 1$, sixth-order Pell-Lucas numbers have the following properties:

- (a) $\sum_{k=1}^n Q_{-k} = \frac{1}{6}(-Q_{-n+5} + Q_{-n+4} + 2Q_{-n+3} + 3Q_{-n+2} + 4Q_{-n+1} + 5Q_{-n} - 14)$.
- (b) $\sum_{k=1}^n Q_{-2k} = \frac{1}{6}(Q_{-2n+4} - 4Q_{-2n+3} + 4Q_{-2n+2} - 3Q_{-2n+1} + 5Q_{-2n} - 2Q_{-2n-1} - 28)$.
- (c) $\sum_{k=1}^n Q_{-2k+1} = \frac{1}{6}(-2Q_{-2n+4} + 5Q_{-2n+3} - 2Q_{-2n+2} + 6Q_{-2n+1} - Q_{-2n} + Q_{-2n-1} + 14)$.

If $r = s = t = 1, u = 1, v = 1, y = 2$ then $(r + s + t + u + v + y - 1)(r - s + t - u + v - y + 1) = 0$ so we can't use Theorem 3.1 (b).

Proposition 3.3. If $r = s = t = 1, u = 1, v = 1, y = 2$ then for $n \geq 1$ we have the following formula:

$$\sum_{k=1}^n W_{-k} = \frac{1}{6}(-W_{-n+5} + W_{-n+3} + 2W_{-n+2} + 3W_{-n+1} + 4W_{-n} + W_5 - W_3 - 2W_2 - 3W_1 - 4W_0)$$

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Proposition, we have the following Corollary which presents linear sum formula of sixth-order Jacobsthal numbers.

Corollary 3.6. For $n \geq 1$, sixth order Jacobsthal numbers have the following property:

$$\sum_{k=1}^n J_{-k} = \frac{1}{6}(J_{-n+3} + 2J_{-n+2} + 3J_{-n+1} + 4J_{-n} - J_{-n+5} - 5)$$

From the last Proposition, we have the following Corollary which gives linear sum formulas of sixth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$).

Corollary 3.7. For $n \geq 1$, sixth order Jacobsthal-Lucas numbers have the following property:

$$\sum_{k=1}^n j_{-k} = \frac{1}{6}(j_{-n+3} + 2j_{-n+2} + 3j_{-n+1} + 4j_{-n} - j_{-n+5} + 9)$$

4 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Hexanacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Hexanacci, Hexanacci-Lucas, sixth order Pell, sixth order Pell-Lucas, sixth order Jacobsthal, sixth order Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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