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On Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of $\sum_{k=0}^{n} kW_k^3$ and $\sum_{k=1}^{n} kW_{-k}^3$

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kW_k^3$ and $\sum_{k=1}^{n} kW_{-k}^3$ for the cubes of generalized Fibonacci numbers are presented. As special cases, we give sum formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Our work generalize second order recurrence relations.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas.

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1 Introduction

There are so many studies in the literature that concern about special second order recurrence sequences such as Fibonacci and Lucas sequences. The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \ F_1 = 1.$$

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and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \ L_1 = 1.$$

Fibonacci and Lucas integer sequences and their generalization/extensions have many interesting, pretty and amazing properties and applications in many fields of science and arts. Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \ W_1 = b, \quad (n \geq 2) \tag{1.1}$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [2,3] and [4]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	Notation: $W_n(a, b; r, s)$	OEIS: [5]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=1}^{n} kP_k^2 = \frac{1}{8}(-P_{n+2}^2 - (9 + 8n)\,P_{n+1}^2 + 2\,(3 + 2n)\,P_{n+2}P_{n+1} + 1)$$

and

$$\sum_{k=1}^{n} kF_{-k}^2 = \frac{1}{2}(-F_{-n+1}^2 + (-1 + 2n)F_{-n}^2 + (1 - 2n)F_{-n+1}F_{-n} + 1).$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[6,7,8,9,10,11]	[12,13,14]	[15,16,17]
Generalized Tribonacci	[18,19]		
Generalized Tetranacci	[20,21]		

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 1.1. *For $n \geq 0$ we have the following formulas: If $(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1) \neq 0$ then*

(a)

$$\sum_{k=0}^{n} W_k^3 = \frac{\Delta_1}{(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_1 &= -(s^3+2rs-1)W_{n+2}^3 - (r^4s+3r^2s^2-r^3s^3+2rs+r^3+s^3-1)W_{n+1}^3 \\ &\quad +3rs(r+s^2)W_{n+2}^2W_{n+1} - 3rs^2(rs-1)W_{n+1}^2W_{n+2} + (2rs+s^3-1)W_1^3 \\ &\quad +(r^4s+3r^2s^2-r^3s^3+2rs+r^3+s^3-1)W_0^3 - 3rs(r+s^2)W_1^2W_0 + 3rs^2(rs-1)W_0^2W_1.\end{aligned}$$

(b)

$$\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{\Delta_2}{(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_2 &= -r(rs-1)W_{n+2}^3 - rs^3(rs-1)W_{n+1}^3 + s(2r^3-s^3+1)W_{n+2}^2W_{n+1} \\ &\quad -(-2rs^4+r^4s+2rs+r^3+s^3-1)W_{n+1}^2W_{n+2} + r(rs-1)W_1^3 \\ &\quad +rs^3(rs-1)W_0^3 - s(2r^3-s^3+1)W_1^2W_0 + (-2rs^4+r^4s+2rs+r^3+s^3-1)W_0^2W_1.\end{aligned}$$

(c)

$$\sum_{k=0}^{n} W_{k+1}^2 W_k = \frac{\Delta_3}{(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_3 &= r(r+s^2)W_{n+2}^3 + rs^3(r+s^2)W_{n+1}^3 - (3r^2s^2+r^3+s^3-1)W_{n+2}^2W_{n+1} \\ &\quad +s^2(2r^3-s^3+1)W_{n+1}^2W_{n+2} - r(r+s^2)W_1^3 - rs^3(r+s^2)W_0^3 \\ &\quad +(3r^2s^2+r^3+s^3-1)W_1^2W_0 - s^2(2r^3-s^3+1)W_0^2W_1.\end{aligned}$$

Proof. This is given in [13].

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 1.2. *For $n \geq 1$ we have the following formulas:If $(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} W_{-k}^3 = \frac{\Delta_4}{(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_4 &= (2rs+s^3-1)W_{-n+1}^3 + (r^4s+3r^2s^2-r^3s^3+2rs+r^3+s^3-1)W_{-n}^3 - 3rs(r+s^2)W_{-n+1}^2W_{-n} \\ &\quad +3rs^2(rs-1)W_{-n+1}W_{-n}^2 - (2rs+s^3-1)W_1^3 \\ &\quad -(r^4s+3r^2s^2-r^3s^3+2rs+r^3+s^3-1)W_0^3 + 3rs(r+s^2)W_1^2W_0 - 3rs^2(rs-1)W_0^2W_1.\end{aligned}$$

(b)

$$\sum_{k=1}^{n} W_{-k+1}^2 W_{-k} = \frac{\Delta_5}{(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_5 &= -r(r+s^2)W_{-n+1}^3 - rs^3(r+s^2)W_{-n}^3 + (3r^2s^2+r^3+s^3-1)W_{-n+1}^2W_{-n} \\ &\quad -s^2(2r^3-s^3+1)W_{-n}^2W_{-n+1} + r(r+s^2)W_1^3 + rs^3(r+s^2)W_0^3 \\ &\quad -(3r^2s^2+r^3+s^3-1)W_1^2W_0 + s^2(2r^3-s^3+1)W_0^2W_1.\end{aligned}$$

(c)

$$\sum_{k=1}^{n} W_{-k}^{2} W_{-k+1} = \frac{\Delta_6}{(rs - s^3 + 1)(r + s - 1)(r + s - rs + r^2 + s^2 + 1)}$$

where

$$\begin{aligned}
\Delta_6 \quad = \quad & r\,(rs-1)\,W_{-n+1}^{3} + rs^{3}\,(rs-1)\,W_{-n}^{3} - s(2r^{3} - s^{3} + 1)W_{-n+1}^{2}W_{-n} \\
& +(-2rs^{4} + r^{4}s + 2rs + r^{3} + s^{3} - 1)W_{-n}^{2}W_{-n+1} - r\,(rs-1)\,W_{1}^{3} \\
& -rs^{3}\,(rs-1)\,W_{0}^{3} + s(2r^{3} - s^{3} + 1)W_{1}^{2}W_{0} - (-2rs^{4} + r^{4}s + 2rs + r^{3} + s^{3} - 1)W_{0}^{2}W_{1}.
\end{aligned}$$

Proof. This is given in [13].

2 Sum Formulas of Generalized Fibonacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 0$ we have the following formulas: If $(rs - s^3 + 1)(r + s - 1)(r + s - rs + r^2 + s^2 + 1) \neq 0$ then*

(a)

$$\sum_{k=0}^{n} kW_{k}^{3} = \frac{\Omega_1}{(rs - s^3 + 1)^2(r + s - 1)^2(r + s - rs + r^2 + s^2 + 1)^2}$$

where

$$\begin{aligned}
\Omega_1 \quad = \quad & (n(-rs + s^{3} - 1)(2rs + s^{3} - 1)(r + s - 1)(r + s - rs + r^{2} + s^{2} + 1) \\
& -8rs^{4} - 4r^{4}s - 8r^{2}s^{2} - 8r^{3}s^{3} - 6r^{2}s^{5} - 2r^{5}s^{2} - 2r^{4}s^{4} + r^{3}s^{6} + 8rs + r^{3} + 4s^{3} - 2s^{6} - 2)W_{n+2}^{3} \\
& +(n(-rs + s^{3} - 1)(r + s - 1)(r + s - rs + r^{2} + s^{2} + 1)(r^{4}s + 3r^{2}s^{2} - r^{3}s^{3} + 2rs + r^{3} + s^{3} - 1) \\
& -2r^{4}s - 4rs^{7} - 2r^{7}s + 2r^{2}s^{2} - 16r^{3}s^{3} - 13r^{2}s^{5} - 10r^{5}s^{2} - 10r^{4}s^{4} + 2r^{3}s^{6} - 4r^{6}s^{3} \\
& -3r^{2}s^{8} + 6r^{5}s^{5} - r^{8}s^{2} - 3r^{4}s^{7} + 2r^{7}s^{4} - r^{6}s^{6} + 4rs + 2r^{3} + s^{3} - r^{6} + s^{6} - s^{9} - 1)W_{n+1}^{3} \\
& +3rs(n(rs - s^{3} + 1)(r + s^{2})(r + s - 1)(r + s - rs + r^{2} + s^{2} + 1) - 3r + 4r^{2}s + 8rs^{3} + r^{5}s \\
& -rs^{6} + 6r^{3}s^{2} + 6r^{2}s^{4} + 2r^{4}s^{3} - r^{3}s^{5} + 2r^{4} - 4s^{2} + 4s^{5})W_{n+2}^{2}W_{n+1} \\
& -3rs^{2}(n(rs - s^{3} + 1)(r + s - 1)(rs - 1)(r + s - rs + r^{2} + s^{2} + 1) \\
& +(r^{2}s^{2} + rs + s^{3} - 1)(-r^{2}s^{2} + 5rs + 2r^{3} - s^{3} - 3))W_{n+1}^{2}W_{n+2} \\
& +(s+1)(-s + s^{2} + 1)(4rs^{4} + 3r^{4}s + 7r^{2}s^{2} - 4rs - 2s^{3} + s^{6} + 1)W_{1}^{3} \\
& +s^{3}(8rs^{4} + 4r^{4}s + 8r^{2}s^{2} + 8r^{3}s^{3} + 6r^{2}s^{5} + 2r^{5}s^{2} + 2r^{4}s^{4} - r^{3}s^{6} - 8rs - r^{3} - 4s^{3} + 2s^{6} + 2)W_{0}^{3} \\
& -3rs(-2r + 2r^{2}s + 4rs^{3} + 2rs^{6} + 2r^{3}s^{2} + 5r^{2}s^{4} + 2r^{4}s^{3} + r^{4} - 3s^{2} + 2s^{5} + s^{8})W_{1}^{2}W_{0} \\
& +3rs^{2}(2r^{4}s + rs^{7} + 4r^{2}s^{2} + 2r^{3}s^{3} + r^{5}s^{2} - 5rs - r^{3} - 2s^{6} + 2)W_{0}^{2}W_{1}
\end{aligned}$$

(b)

$$\sum_{k=0}^{n} kW_{k}^{2}W_{k+1} = \frac{\Omega_2}{(rs - s^3 + 1)^2(r + s - 1)^2(r + s - rs + r^2 + s^2 + 1)^2}$$

where

$$\begin{aligned}
\Omega_2 \;=\;& r(n(-rs+s^3-1)(rs-1)(r+s-1)(r+s-rs+r^2+s^2+1) \\
& -2r^4s-rs^7-4r^2s^2-2r^3s^3-r^5s^2+5rs+r^3+2s^6-2)W_{n+2}^3 \\
& +rs^3(n(-rs+s^3-1)(r+s-1)(rs-1)(r+s-rs+r^2+s^2+1) \\
& -4rs^4-2r^4s-3r^2s^2-6r^3s^3+2r^2s^5-2r^5s^2+r^4s^4+8rs+2r^3+2s^3+s^6-3)W_{n+1}^3 \\
& +s(n(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)(2r^3-s^3+1) \\
& -4rs^4+8r^4s+2rs^7+2r^7s+2r^3s^3-6r^2s^5+6r^5s^2-2r^4s^4 \\
& +3r^3s^6+2rs-5r^3+4s^3+4r^6-2s^6-2)W_{n+2}^2W_{n+1} \\
& +(n(-rs+s^3-1)(r+s-1)(r+s-rs+r^2+s^2+1)(-2rs^4+r^4s+2rs+r^3+s^3-1) \\
& -8rs^4-2r^4s+4rs^7-2r^7s-7r^2s^2+5r^2s^5-4r^5s^2-r^4s^4+10r^3s^6-4r^6s^3-4r^2s^8 \\
& +4r^5s^5-r^8s^2-3r^4s^7+4rs+2r^3+s^3-r^6+s^6-s^9-1)W_{n+1}^2W_{n+2} \\
& +r(-4rs^4+2r^4s+2rs^7+5r^2s^2-2r^3s^3+2r^2s^5+r^4s^4-2rs+2s^3-3s^6+1)W_1^3 \\
& +rs^3(2r^4s+rs^7+4r^2s^2+2r^3s^3+r^5s^2-5rs-r^3-2s^6+2)W_0^3 \\
& +s\,(s+1)\,(-s+s^2+1)(-3r^4s+3r^2s^2-4r^3s^3+4r^3-2s^3-2r^6+s^6+1)W_1^2W_0 \\
& +s^2(rs-1)(-2s+4r^3s+r^6s+2r^3s^4-6r^2+3r^5+4s^4-2s^7)W_0^2W_1
\end{aligned}$$

(c)

$$\sum_{k=0}^{n} kW_{k+1}^2W_k = \frac{\Omega_3}{(rs-s^3+1)^2(r+s-1)^2(r+s-rs+r^2+s^2+1)^2}$$

where

$$\begin{aligned}
\Omega_3 \;=\;& r(n(rs-s^3+1)(r+s-1)(r+s^2)(r+s-rs+r^2+s^2+1) \\
& +2r^2s+4rs^3+2rs^6+2r^3s^2+5r^2s^4+2r^4s^3+r^4-3s^2+2s^5+s^8-2r)W_{n+2}^3 \\
& +rs^3(n(r+s^2)(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1) \\
& +4r^2s+8rs^3+r^5s-rs^6+6r^3s^2+6r^2s^4+2r^4s^3-r^3s^5+2r^4-4s^2+4s^5-3r)W_{n+1}^3 \\
& +(n(-rs+s^3-1)(r+s-1)(r+s-rs+r^2+s^2+1)(3r^2s^2+r^3+s^3-1) \\
& -3r^4s+6r^2s^2-12r^3s^3-9r^2s^5-6r^5s^2-12r^4s^4 \\
& -2r^3s^6-4r^6s^3-3r^2s^8+2r^3+s^3-r^6+s^6-s^9-1)W_{n+2}^2W_{n+1} \\
& +s^2(n(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1)(2r^3-s^3+1) \\
& -4rs^4+8r^4s+2rs^7+2r^7s+2r^3s^3-6r^2s^5+6r^5s^2-2r^4s^4+3r^3s^6 \\
& +2rs-5r^3+4s^3+4r^6-2s^6-2)W_{n+1}^2W_{n+2} \\
& +r(r+r^5s-5rs^6+2r^3s^2-4r^2s^4-2r^4s^3-r^3s^5+2s^2-2s^8)W_1^3 \\
& -rs^3(-2r+2r^2s+4rs^3+2rs^6+2r^3s^2+5r^2s^4+2r^4s^3+r^4-3s^2+2s^5+s^8)W_0^3 \\
& +s(r+s^2)(4r^3s^3+6r^2s^5+3r^5s^2+2r^3-4s^3-r^6+2s^6+2)W_1^2W_0 \\
& +s^2(s+1)(-s+s^2+1)(-3r^4s+3r^2s^2-4r^3s^3+4r^3-2s^3-2r^6+s^6+1)W_0^2W_1
\end{aligned}$$

Proof. Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$s^3W_n^3 = W_{n+2}^3 - 3rW_{n+2}^2W_{n+1} + 3r^2W_{n+1}^2W_{n+2} - r^3W_{n+1}^3$$

and so

$$\begin{aligned}
s^3 \times n \times W_n^3 &= n \times W_{n+2}^3 - 3r \times n \times W_{n+2}^2 W_{n+1} + 3r^2 \times n \times W_{n+1}^2 W_{n+2} - r^3 \times n \times W_{n+1}^3 \\
s^3(n-1)W_{n-1}^3 &= (n-1)W_{n+1}^3 - 3r(n-1)W_{n+1}^2 W_n + 3r^2(n-1)W_n^2 W_{n+1} - r^3(n-1)W_n^3 \\
s^3(n-2)W_{n-2}^3 &= (n-2)W_n^3 - 3r(n-2)W_n^2 W_{n-1} + 3r^2(n-2)W_{n-1}^2 W_n - r^3(n-2)W_{n-1}^3 \\
&\vdots \\
s^3 \times 2 \times W_2^3 &= 2 \times W_4^3 - 3r \times 2 \times W_4^2 W_3 + 3r^2 \times 2 \times W_3^2 W_4 - r^3 \times 2 \times W_3^3 \\
s^3 \times 1 \times W_1^3 &= 1 \times W_3^3 - 3r \times 1 \times W_3^2 W_2 + 3r^2 \times 1 \times W_2^2 W_3 - r^3 \times 1 \times W_2^3 \\
s^3 \times 0 \times W_0^3 &= 0 \times W_2^3 - 3r \times 0 \times W_2^2 W_1 + 3r^2 \times 0 \times W_1^2 W_2 - r^3 \times 0 \times W_1^3
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s^3 \sum_{k=0}^{n} kW_k^3 &= (nW_{n+2}^3 + (n-1)W_{n+1}^3 - (-1)W_1^3 - (-2)W_0^3 + \sum_{k=0}^{n} kW_k^3 - 2\sum_{k=0}^{n} W_k^3) \quad (2.1) \\
&\quad -3r(nW_{n+2}^2 W_{n+1} - (-1)W_1^2 W_0 + \sum_{k=0}^{n} kW_{k+1}^2 W_k - \sum_{k=0}^{n} W_{k+1}^2 W_k) \\
&\quad +3r^2(nW_{n+1}^2 W_{n+2} - (-1)W_0^2 W_1 + \sum_{k=0}^{n} kW_k^2 W_{k+1} - \sum_{k=0}^{n} W_k^2 W_{k+1}) \\
&\quad -r^3(nW_{n+1}^3 - (-1)W_0^3 + \sum_{k=0}^{n} kW_k^3 - \sum_{k=0}^{n} W_k^3).
\end{aligned}$$

Next we calculate $\sum_{k=0}^{n} kW_{k+1}^2 W_k$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$sW_{n+1}^2 W_n = W_{n+1}^2 W_{n+2} - rW_{n+1}^3$$

and so

$$\begin{aligned}
s \times n \times W_{n+1}^2 W_n &= nW_{n+1}^2 W_{n+2} - r \times n \times W_{n+1}^3 \\
s(n-1)W_n^2 W_{n-1} &= (n-1)W_n^2 W_{n+1} - r(n-1)W_n^3 \\
s(n-2)W_{n-1}^2 W_{n-2} &= (n-2)W_{n-1}^2 W_n - r(n-2)W_{n-1}^3 \\
&\vdots \\
s \times 2 \times W_3^2 W_2 &= 2 \times W_3^2 W_4 - r \times 2 \times W_3^3 \\
s \times 1 \times W_2^2 W_1 &= 1 \times W_2^2 W_3 - r \times 1 \times W_2^3 \\
s \times 0 \times W_1^2 W_0 &= 0 \times W_1^2 W_2 - r \times 0 \times W_1^3
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s \sum_{k=0}^{n} kW_{k+1}^2 W_k &= (nW_{n+1}^2 W_{n+2} - (-1)W_0^2 W_1 + \sum_{k=0}^{n} kW_k^2 W_{k+1} - \sum_{k=0}^{n} W_k^2 W_{k+1}) \quad (2.2) \\
&\quad -r(nW_{n+1}^3 - (-1)W_0^3 + \sum_{k=0}^{n} kW_k^3 - \sum_{k=0}^{n} W_k^3).
\end{aligned}$$

Next we calculate $\sum_{k=0}^{n} kW_k^2 W_{k+1}$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1} \Rightarrow s^2 W_n^2 = W_{n+2}^2 + r^2 W_{n+1}^2 - 2rW_{n+2}W_{n+1}$$

we obtain

$$s^2 W_n^2 W_{n+1} = W_{n+2}^2 W_{n+1} + r^2 W_{n+1}^3 - 2rW_{n+1}^2 W_{n+2}$$

and so

$$\begin{aligned}
s^2 \times n \times W_n^2 W_{n+1} &= n \times W_{n+2}^2 W_{n+1} + r^2 \times n \times W_{n+1}^3 - 2r \times n \times W_{n+1}^2 W_{n+2} \\
s^2 (n-1) W_{n-1}^2 W_n &= (n-1) W_{n+1}^2 W_n + r^2 (n-1) W_n^3 - 2r(n-1) W_n^2 W_{n+1} \\
s^2 (n-2) W_{n-2}^2 W_{n-1} &= (n-2) W_n^2 W_{n-1} + r^2 (n-2) W_{n-1}^3 - 2r(n-2) W_{n-1}^2 W_n \\
&\vdots \\
s^2 \times 2 \times W_2^2 W_3 &= 2 \times W_4^2 W_3 + r^2 \times 2 \times W_3^3 - 2r \times 2 \times W_3^2 W_4 \\
s^2 \times 1 \times W_1^2 W_2 &= 1 \times W_3^2 W_2 + r^2 \times 1 \times W_2^3 - 2r \times 1 \times W_2^2 W_3 \\
s^2 \times 0 \times W_0^2 W_1 &= 0 \times W_2^2 W_1 + r^2 \times 0 \times W_1^3 - 2r \times 0 \times W_1^2 W_2
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s^2 \sum_{k=0}^{n} kW_k^2 W_{k+1} &= (nW_{n+2}^2 W_{n+1} - (-1)W_1^2 W_0 + \sum_{k=0}^{n} kW_{k+1}^2 W_k - \sum_{k=0}^{n} W_{k+1}^2 W_k) \qquad (2.3) \\
&\quad + r^2 (nW_{n+1}^3 - (-1)W_0^3 + \sum_{k=0}^{n} kW_k^3 - \sum_{k=0}^{n} W_k^3) \\
&\quad - 2r(nW_{n+1}^2 W_{n+2} - (-1)W_0^2 W_1 + \sum_{k=0}^{n} kW_k^2 W_{k+1} - \sum_{k=0}^{n} W_k^2 W_{k+1}).
\end{aligned}$$

Using Theorem 1.1 and solving the system (2.1)-(2.2)-(2.3), the required results of (a),(b) and (c) follow.

Taking $r = s = 1$ in Theorem 2.1 (a),(b) and (c), we obtain the following proposition.

Proposition 2.1. *If $r = s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k^3 = \frac{1}{4}(-(2n+7)W_{n+2}^3 - (6n+13)W_{n+1}^3 + 3(2n+6)W_{n+2}^2 W_{n+1} - 3W_{n+1}^2 W_{n+2} + 5W_1^3 + 7W_0^3 - 12W_1^2 W_0 + 3W_0^2 W_1)$.

(b) $\sum_{k=0}^{n} kW_k^2 W_{k+1} = \frac{1}{4}(-W_{n+2}^3 - W_{n+1}^3 + (2n+3)W_{n+2}^2 W_{n+1} - 2(n+1)W_{n+1}^2 W_{n+2} - W_1^2 W_0 + W_1^3 + W_0^3)$.

(c) $\sum_{k=0}^{n} kW_{k+1}^2 W_k = \frac{1}{4}(2(n+2)W_{n+2}^3 + 2(n+3)W_{n+1}^3 + (2n+3)W_{n+1}^2 W_{n+2} - (4n+11)W_{n+2}^2 W_{n+1} - 2W_1^3 - 4W_0^3 - W_0^2 W_1 + 7W_1^2 W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.2. *For $n \geq 0$, Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} kF_k^3 = \frac{1}{4}(-(2n+7)F_{n+2}^3 - (6n+13)F_{n+1}^3 + 3(2n+6)F_{n+2}^2 F_{n+1} - 3F_{n+1}^2 F_{n+2} + 5)$.

(b) $\sum_{k=0}^{n} kF_k^2 F_{k+1} = \frac{1}{4}(-F_{n+2}^3 - F_{n+1}^3 + (2n+3)F_{n+2}^2 F_{n+1} - 2(n+1)F_{n+1}^2 F_{n+2} + 1)$.

(c) $\sum_{k=0}^{n} kF_{k+1}^2F_k = \frac{1}{4}(2(n+2)F_{n+2}^3+2(n+3)F_{n+1}^3+(2n+3)F_{n+1}^2F_{n+2}-(4n+11)F_{n+2}^2F_{n+1}-2)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.3. *For $n \geq 0$, Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kL_k^3 = \frac{1}{4}(-(2n+7)L_{n+2}^3-(6n+13)L_{n+1}^3+3(2n+6)L_{n+2}^2L_{n+1}-3L_{n+1}^2L_{n+2}+49)$.

(b) $\sum_{k=0}^{n} kL_k^2L_{k+1} = \frac{1}{4}(-L_{n+2}^3-L_{n+1}^3+(2n+3)L_{n+2}^2L_{n+1}-2(n+1)L_{n+1}^2L_{n+2}+7)$.

(c) $\sum_{k=0}^{n} kL_{k+1}^2L_k = \frac{1}{4}(2(n+2)L_{n+2}^3+2(n+3)L_{n+1}^3+(2n+3)L_{n+1}^2L_{n+2}-(4n+11)L_{n+2}^2L_{n+1}-24)$.

Taking $r = 2, s = 1$ in Theorem 2.1 (a),(b) and (c), we obtain the following proposition.

Proposition 2.2. *If $r = 2, s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k^3 = \frac{1}{98}(-(14n+33)W_{n+2}^3-(112n+145)W_{n+1}^3+3(21n+46)W_{n+2}^2W_{n+1}-3(7n+27)W_{n+1}^2W_{n+2}+19W_1^3+33W_0^3-75W_1^2W_0+60W_0^2W_1)$.

(b) $\sum_{k=0}^{n} kW_k^2W_{k+1} = \frac{1}{98}(-(7n+20)W_{n+2}^3-(7n+27)W_{n+1}^3+(56n+97)W_{n+2}^2W_{n+1}-(84n+107)W_{n+1}^2W_{n+2}+13W_1^3+20W_0^3-41W_1^2W_0+23W_0^2W_1)$.

(c) $\sum_{k=0}^{n} kW_{k+1}^2W_k = \frac{1}{98}((21n+25)W_{n+2}^3+(21n+46)W_{n+1}^3-(70n+109)W_{n+2}^2W_{n+1}+(56n+97)W_{n+1}^2W_{n+2}-4W_1^3-25W_0^3+39W_1^2W_0-41W_0^2W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.4. *For $n \geq 0$, Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} kP_k^3 = \frac{1}{98}(-(14n+33)P_{n+2}^3-(112n+145)P_{n+1}^3+3(21n+46)P_{n+2}^2P_{n+1}-3(7n+27)P_{n+1}^2P_{n+2}+19)$.

(b) $\sum_{k=0}^{n} kP_k^2P_{k+1} = \frac{1}{98}(-(7n+20)P_{n+2}^3-(7n+27)P_{n+1}^3+(56n+97)P_{n+2}^2P_{n+1}-(84n+107)P_{n+1}^2P_{n+2}+13)$.

(c) $\sum_{k=0}^{n} kP_{k+1}^2P_k = \frac{1}{98}((21n+25)P_{n+2}^3+(21n+46)P_{n+1}^3-(70n+109)P_{n+2}^2P_{n+1}+(56n+97)P_{n+1}^2P_{n+2}-4)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.5. *For $n \geq 0$, Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kQ_k^3 = \frac{1}{98}(-(14n+33)Q_{n+2}^3-(112n+145)Q_{n+1}^3+3(21n+46)Q_{n+2}^2Q_{n+1}-3(7n+27)Q_{n+1}^2Q_{n+2}+296)$.

(b) $\sum_{k=0}^{n} kQ_k^2Q_{k+1} = \frac{1}{98}(-(7n+20)Q_{n+2}^3-(7n+27)Q_{n+1}^3+(56n+97)Q_{n+2}^2Q_{n+1}-(84n+107)Q_{n+1}^2Q_{n+2}+120)$.

(c) $\sum_{k=0}^{n} kQ_{k+1}^2Q_k = \frac{1}{98}((21n+25)Q_{n+2}^3+(21n+46)Q_{n+1}^3-(70n+109)Q_{n+2}^2Q_{n+1}+(56n+97)Q_{n+1}^2Q_{n+2}-248)$.

Taking $r = 1, s = 2$ in Theorem 2.1 (a),(b) and (c), we obtain the following proposition.

Proposition 2.3. *If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k^3 = \frac{1}{4900}((770n-481)W_{n+2}^3+(1260n-2588)W_{n+1}^3-6(350n-225)W_{n+2}^2W_{n+1}+12(70n+39)W_{n+1}^2W_{n+2}+1251W_1^3+3848W_0^3-3450W_1^2W_0+372W_0^2W_1)$.

(b) $\sum_{k=0}^{n} kW_k^2W_{k+1} = \frac{1}{4900}((70n-31)W_{n+2}^3 + 8(70n+39)W_{n+1}^3 + 2(350n+125)W_{n+2}^2W_{n+1} - (1260n+632)W_{n+1}^2W_{n+2} + 101W_1^3 + 248W_0^3 + 450W_1^2W_0 - 628W_0^2W_1)$.

(c) $\sum_{k=0}^{n} kW_{k+1}^2W_k = \frac{1}{4900}(-(350n-575)W_{n+2}^3 - 8(350n-225)W_{n+1}^3 + (1400n-1950)W_{n+2}^2 W_{n+1} + 4(350n+125)W_{n+1}^2W_{n+2} - 925W_1^3 - 4600W_0^3 + 3350W_1^2W_0 + 900W_0^2W_1)$.

From the last proposition we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.6. *For $n \geq 0$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} kJ_k^3 = \frac{1}{4900}((770n-481)J_{n+2}^3 + (1260n-2588)J_{n+1}^3 - 6(350n-225)J_{n+2}^2J_{n+1} + 12(70n+39)J_{n+1}^2J_{n+2} + 1251)$.

(b) $\sum_{k=0}^{n} kJ_k^2J_{k+1} = \frac{1}{4900}((70n-31)J_{n+2}^3 + 8(70n+39)J_{n+1}^3 + 2(350n+125)J_{n+2}^2J_{n+1} - (1260n+632)J_{n+1}^2J_{n+2} + 101)$.

(c) $\sum_{k=0}^{n} kJ_{k+1}^2J_k = \frac{1}{4900}(-(350n-575)J_{n+2}^3 - 8(350n-225)J_{n+1}^3 + (1400n-1950)J_{n+2}^2J_{n+1} + 4(350n+125)J_{n+1}^2J_{n+2} - 925)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.7. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kj_k^3 = \frac{1}{4900}((770n-481)j_{n+2}^3 + (1260n-2588)j_{n+1}^3 - 6(350n-225)j_{n+2}^2j_{n+1} + 12(70n+39)j_{n+1}^2j_{n+2} + 26623)$.

(b) $\sum_{k=0}^{n} kj_k^2j_{k+1} = \frac{1}{4900}((70n-31)j_{n+2}^3 + 8(70n+39)j_{n+1}^3 + 2(350n+125)j_{n+2}^2j_{n+1} - (1260n+632)j_{n+1}^2j_{n+2} + 473)$.

(c) $\sum_{k=0}^{n} kj_{k+1}^2j_k = \frac{1}{4900}(-(350n-575)j_{n+2}^3 - 8(350n-225)j_{n+1}^3 + (1400n-1950)j_{n+2}^2j_{n+1} + 4(350n+125)j_{n+1}^2j_{n+2} - 27425)$.

3 Sum Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas:If $(rs - s^3 + 1)(r + s - 1)(r + s - rs + r^2 + s^2 + 1) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} kW_{-k}^3 = \frac{\Omega_4}{(rs - s^3 + 1)^2(r + s - 1)^2(r + s - rs + r^2 + s^2 + 1)^2}$$

where

$$\begin{aligned}
\Omega_4 = {} & (n(2rs+s^3-1)(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1) \\
& -3r^4s-4rs^7-7r^2s^2-7r^2s^5-3r^4s^4+4rs+s^3+s^6-s^9-1)W_{-n+1}^3 \\
& +(n(r+s-1)(rs-s^3+1)(r^4s+3r^2s^2-r^3s^3+2rs+r^3+s^3-1)(r+s-rs+r^2+s^2+1) \\
& +8rs^4-8rs^7+r^3s^3-8r^2s^5-4r^4s^4-8r^3s^6-6r^2s^8 \\
& -2r^5s^5-2r^4s^7+r^3s^9-2s^3+4s^6-2s^9)W_{-n}^3 \\
& -3rs(n(r+s^2)(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1) \\
& +2r-2r^2s-4rs^3-2rs^6-2r^3s^2-5r^2s^4-2r^4s^3-r^4+3s^2-2s^5-s^8)W_{-n+1}^2W_{-n} \\
& +3rs^2(n(rs-s^3+1)(r+s-1)(rs-1)(r+s-rs+r^2+s^2+1) \\
& -2r^4s-rs^7-4r^2s^2-2r^3s^3-r^5s^2+5rs+r^3+2s^6-2)W_{-n}^2W_{-n+1} \\
& +(s+1)(-s+s^2+1)(4rs^4+3r^4s+7r^2s^2-4rs-2s^3+s^6+1)W_1^3 \\
& +s^3(8rs^4+4r^4s+8r^2s^2+8r^3s^3+6r^2s^5+2r^5s^2+2r^4s^4-r^3s^6-8rs-r^3-4s^3+2s^6+2)W_0^3 \\
& -3rs(-2r+2r^2s+4rs^3+2rs^6+2r^3s^2+5r^2s^4+2r^4s^3+r^4-3s^2+2s^5+s^8)W_1^2W_0 \\
& +3rs^2(2r^4s+rs^7+4r^2s^2+2r^3s^3+r^5s^2-5rs-r^3-2s^6+2)W_0^2W_1
\end{aligned}$$

(b)

$$\sum_{k=1}^{n} kW_{-k+1}^2W_{-k} = \frac{\Omega_5}{(rs-s^3+1)^2(r+s-1)^2(r+s-rs+r^2+s^2+1)^2}$$

where

$$\begin{aligned}
\Omega_5 = {} & -r(n(r+s^2)(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1) \\
& +r+r^5s-5rs^6+2r^3s^2-4r^2s^4-2r^4s^3-r^3s^5+2s^2-2s^8)W_{-n+1}^3 \\
& +rs^3(n(-rs+s^3-1)(r+s^2)(r+s-1)(r+s-rs+r^2+s^2+1) \\
& +2r^2s+4rs^3+2rs^6+2r^3s^2+5r^2s^4+2r^4s^3+r^4-3s^2+2s^5+s^8-2r)W_{-n}^3 \\
& +(n(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1)(3r^2s^2+r^3+s^3-1) \\
& +4rs^4-2r^4s-2rs^7+r^7s-2r^3s^3-4r^4s^4-10r^3s^6-2r^6s^3 \\
& -6r^2s^8-3r^5s^5-2rs-2s^3+4s^6-2s^9)W_{-n+1}^2W_{-n} \\
& +s^2(n(r+s-1)(-rs+s^3-1)(r+s-rs+r^2+s^2+1)(2r^3-s^3+1) \\
& +3r^4s-3r^2s^2-3r^2s^5+3r^4s^4+4r^3s^6+2r^6s^3-4r^3+s^3+2r^6+s^6-s^9-1)W_{-n}^2W_{-n+1} \\
& +r(r+r^5s-5rs^6+2r^3s^2-4r^2s^4-2r^4s^3-r^3s^5+2s^2-2s^8)W_1^3 \\
& -rs^3(-2r+2r^2s+4rs^3+2rs^6+2r^3s^2+5r^2s^4+2r^4s^3+r^4-3s^2+2s^5+s^8)W_0^3 \\
& +s(r+s^2)(4r^3s^3+6r^2s^5+3r^5s^2+2r^3-4s^3-r^6+2s^6+2)W_1^2W_0 \\
& +s^2\,(s+1)\,(-s+s^2+1)(-3r^4s+3r^2s^2-4r^3s^3+4r^3-2s^3-2r^6+s^6+1)W_0^2W_1
\end{aligned}$$

(c)

$$\sum_{k=1}^{n} kW_{-k}^2W_{-k+1} = \frac{\Omega_6}{(rs-s^3+1)^2(r+s-1)^2(r+s-rs+r^2+s^2+1)^2}$$

where

$$\begin{aligned}
\Omega_6 = {} & r(n(r+s-1)(rs-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)+4rs^4-2r^4s-2rs^7 \\
& -5r^2s^2+2r^3s^3-2r^2s^5-r^4s^4+2rs-2s^3+3s^6-1)W_{-n+1}^3 \\
& +rs^3(n(r+s-1)(rs-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1) \\
& -2r^4s-rs^7-4r^2s^2-2r^3s^3-r^5s^2+5rs+r^3+2s^6-2)W_{-n}^3 \\
& +s(n(-rs+s^3-1)(r+s-1)(r+s-rs+r^2+s^2+1)(2r^3-s^3+1) \\
& +3r^4s-3r^2s^2-3r^2s^5+3r^4s^4+4r^3s^6+2r^6s^3-4r^3+s^3+2r^6+s^6-s^9-1)W_{-n+1}^2W_{-n} \\
& +(n(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)(-2rs^4+r^4s+2rs+r^3+s^3-1) \\
& +10r^3s^3+3r^5s^2-4r^4s^4+2r^3s^6-2r^6s^3-2r^4s^7-r^7s^4 \\
& -2s^3+4s^6-2s^9+2rs^4-4rs^7+2rs^{10}-6r^2s^2)W_{-n}^2W_{-n+1} \\
& +r(-4rs^4+2r^4s+2rs^7+5r^2s^2-2r^3s^3+2r^2s^5+r^4s^4-2rs+2s^3-3s^6+1)W_1^3 \\
& +rs^3(2r^4s+rs^7+4r^2s^2+2r^3s^3+r^5s^2-5rs-r^3-2s^6+2)W_0^3 \\
& +s(s+1)(-s+s^2+1)(-3r^4s+3r^2s^2-4r^3s^3+4r^3-2s^3-2r^6+s^6+1)W_1^2W_0 \\
& +s^2(rs-1)(-2s+4r^3s+r^6s+2r^3s^4-6r^2+3r^5+4s^4-2s^7)W_1W_0^2
\end{aligned}$$

Proof. Using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$s^3W^3_{-n} = W^3_{-n+2} - 3rW^2_{-n+2}W_{-n+1} + 3r^2W^2_{-n+1}W_{-n+2} - r^3W^3_{-n+1}$$

and so

$$\begin{aligned}
s^3 \times n \times W^3_{-n} &= nW^3_{-n+2} - 3r \times n \times W^2_{-n+2}W_{-n+1} + 3r^2 \times n \times W^2_{-n+1}W_{-n+2} - r^3 \times n \times W^3_{-n+1} \\
s^3(n-1)W^3_{-n+1} &= (n-1)W^3_{-n+3} - 3r(n-1)W^2_{-n+3}W_{-n+2} + 3r^2(n-1)W^2_{-n+2}W_{-n+3} - r^3(n-1)W^3_{-n+2} \\
s^3(n-2)W^3_{-n+2} &= (n-2)W^3_{-n+4} - 3r(n-2)W^2_{-n+4}W_{-n+3} + 3r^2(n-2)W^2_{-n+3}W_{-n+4} - r^3(n-2)W^3_{-n+3} \\
&\vdots \\
s^3 \times 3 \times W^3_{-3} &= 3 \times W^3_{-1} - 3r \times 3 \times W^2_{-1}W_{-2} + 3r^2 \times 3 \times W^2_{-2}W_{-1} - r^3 \times 3 \times W^3_{-2} \\
s^3 \times 2 \times W^3_{-2} &= 2 \times W^3_0 - 3r \times 2 \times W^2_0W_{-1} + 3r^2 \times 2 \times W^2_{-1}W_0 - r^3 \times 2 \times W^3_{-1} \\
s^3 \times 1 \times W^3_{-1} &= 1 \times W^3_1 - 3r \times 1 \times W^2_1W_0 + 3r^2 \times 1 \times W^2_0W_1 - r^3 \times 1 \times W^3_0
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s^3(\sum_{k=1}^{n} kW^3_{-k}) &= ((-n-1)W^3_{-n+1} + (-n-2)W^3_{-n} + W^3_1 + 2 \times W^3_0 + \sum_{k=1}^{n} kW^3_{-k} + 2\sum_{k=1}^{n} W^3_{-k}) \\
&\quad -3r((-n-1)W^2_{-n+1}W_{-n} + W^2_1W_0 + \sum_{k=1}^{n} kW^2_{-k+1}W_{-k} + \sum_{k=1}^{n} W^2_{-k+1}W_{-k}) \\
&\quad +3r^2((-n-1)W^2_{-n}W_{-n+1} + W^2_0W_1 + \sum_{k=1}^{n} kW^2_{-k}W_{-k+1} + \sum_{k=1}^{n} W^2_{-k}W_{-k+1}) \\
&\quad -r^3((-n-1)W^3_{-n} + W^3_0 + \sum_{k=1}^{n} kW^3_{-k} + \sum_{k=1}^{n} W^3_{-k})
\end{aligned} \tag{3.1}$$

Next we calculate $\sum_{k=1}^{n} kW^2_{-k+1}W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$sW^2_{-n+1}W_{-n} = W^2_{-n+1}W_{-n+2} - rW^3_{-n+1}$$

and so

$$\begin{aligned}
s \times n \times W^2_{-n+1}W_{-n} &= nW^2_{-n+1}W_{-n+2} - r \times n \times W^3_{-n+1} \\
s(n-1)W^2_{-n+2}W_{-n+1} &= (n-1)W^2_{-n+2}W_{-n+3} - r(n-1)W^3_{-n+2} \\
s(n-2)W^2_{-n+3}W_{-n+2} &= (n-2)W^2_{-n+3}W_{-n+4} - r(n-2)W^3_{-n+3} \\
&\vdots \\
s \times 3 \times W^2_{-2}W_{-3} &= 3W^2_{-2}W_{-1} - r \times 3 \times W^3_{-2} \\
s \times 2 \times W^2_{-1}W_{-2} &= 2 \times W^2_{-1}W_0 - r \times 2 \times W^3_{-1} \\
s \times 1 \times W^2_0W_{-1} &= 1 \times W^2_0W_1 - r \times 1 \times W^3_0
\end{aligned}$$

If we add the above equations by side by, we get

$$
\begin{aligned}
s\sum_{k=1}^{n}kW_{-k+1}^{2}W_{-k} &= (-(n+1)W_{-n}^{2}W_{-n+1}+W_{0}^{2}W_{1}+\sum_{k=1}^{n}kW_{-k}^{2}W_{-k+1}+\sum_{k=1}^{n}W_{-k}^{2}W_{-k+1}) \\
&\quad -r(-(n+1)W_{-n}^{3}+W_{0}^{3}+\sum_{k=1}^{n}kW_{-k}^{3}+\sum_{k=1}^{n}W_{-k}^{3})
\end{aligned} \tag{3.2}
$$

Next we calculate $\sum_{k=1}^{n}W_{-k+1}^{2}W_{-k}$. Again using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

we obtain

$$
\begin{aligned}
s^{2}W_{-n}^{2} &= W_{-n+2}^{2}-2rW_{-n+2}W_{-n+1}+r^{2}W_{-n+1}^{2} \\
&\Rightarrow s^{2}W_{-n}^{2}W_{-n+1}=W_{-n+2}^{2}W_{-n+1}-2rW_{-n+1}^{2}W_{-n+2}+r^{2}W_{-n+1}^{3}
\end{aligned}
$$

and so

$$
\begin{aligned}
s^{2}\times n\times W_{-n}^{2}W_{-n+1} &= nW_{-n+2}^{2}W_{-n+1}-2r\times n\times W_{-n+1}^{2}W_{-n+2}+r^{2}\times n\times W_{-n+1}^{3} \\
s^{2}(n-1)W_{-n+1}^{2}W_{-n+2} &= (n-1)W_{-n+3}^{2}W_{-n+2}-2r(n-1)W_{-n+2}^{2}W_{-n+3}+r^{2}(n-1)W_{-n+2}^{3} \\
s^{2}(n-2)W_{-n+2}^{2}W_{-n+3} &= (n-2)W_{-n+4}^{2}W_{-n+3}-2r(n-2)W_{-n+3}^{2}W_{-n+4}+r^{2}(n-2)W_{-n+3}^{3} \\
&\vdots \\
s^{2}\times 3\times W_{-3}^{2}W_{-2} &= 3\times W_{-1}^{2}W_{-2}-2r\times 3\times W_{-2}^{2}W_{-1}+r^{2}\times 3\times W_{-2}^{3} \\
s^{2}\times 2\times W_{-2}^{2}W_{-1} &= 2\times W_{0}^{2}W_{-1}-2r\times 2\times W_{-1}^{2}W_{0}+r^{2}\times 2\times W_{-1}^{3} \\
s^{2}\times 1\times W_{-1}^{2}W_{0} &= 1\times W_{1}^{2}W_{0}-2r\times 1\times W_{0}^{2}W_{1}+r^{2}\times 1\times W_{0}^{3}
\end{aligned}
$$

If we add the above equations by side by, we get

$$
\begin{aligned}
s^{2}\sum_{k=1}^{n}kW_{-k}^{2}W_{-k+1} &= (-(n+1)W_{-n+1}^{2}W_{-n}+W_{1}^{2}W_{0}+\sum_{k=1}^{n}kW_{-k+1}^{2}W_{-k}+\sum_{k=1}^{n}W_{-k+1}^{2}W_{-k}) \\
&\quad -2r(-(n+1)W_{-n}^{2}W_{-n+1}+W_{0}^{2}W_{1}+\sum_{k=1}^{n}kW_{-k}^{2}W_{-k+1}+\sum_{k=1}^{n}W_{-k}^{2}W_{-k+1}) \\
&\quad +r^{2}(-(n+1)W_{-n}^{3}+W_{0}^{3}+\sum_{k=1}^{n}kW_{-k}^{3}+\sum_{k=1}^{n}W_{-k}^{3})
\end{aligned} \tag{3.3}
$$

Then, using Theorem 1.2 and solving the system (3.1)-(3.2)-(3.3), the required results of (a),(b) and (c) follow.

Taking $r=s=1$ in Theorem 3.1 (a),(b) and (c), we obtain the following proposition.

Proposition 3.1. *If $r=s=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}kW_{-k}^{3}=\frac{1}{4}((2n-5)W_{-n+1}^{3}+(6n-7)W_{-n}^{3}-3(2n-4)W_{-n+1}^{2}W_{-n}-3W_{-n}^{2}W_{-n+1}+5W_{1}^{3}+7W_{0}^{3}-12W_{1}^{2}W_{0}+3W_{0}^{2}W_{1})$.

(b) $\sum_{k=1}^{n}kW_{-k+1}^{2}W_{-k}=\frac{1}{4}(2(1-n)W_{-n+1}^{3}+2(2-n)W_{-n}^{3}+(4n-7)W_{-n+1}^{2}W_{-n}+(1-2n)W_{-n}^{2}W_{-n+1}-2W_{1}^{3}-4W_{0}^{3}-W_{0}^{2}W_{1}+7W_{0}W_{1}^{2})$.

(c) $\sum_{k=1}^{n}kW_{-k}^{2}W_{-k+1}=\frac{1}{4}(-W_{-n+1}^{3}-W_{-n}^{3}+(1-2n)W_{-n+1}^{2}W_{-n}+2nW_{-n}^{2}W_{-n+1}+W_{1}^{3}+W_{0}^{3}-W_{1}^{2}W_{0})$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.2. *For $n \geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} kF_{-k}^3 = \frac{1}{4}((2n-5)F_{-n+1}^3 + (6n-7)F_{-n}^3 - 3(2n-4)F_{-n+1}^2 F_{-n} - 3F_{-n}^2 F_{-n+1} + 5)$.

(b) $\sum_{k=1}^{n} kF_{-k+1}^2 F_{-k} = \frac{1}{4}(2(1-n)F_{-n+1}^3 + 2(2-n)F_{-n}^3 + (4n-7)F_{-n+1}^2 F_{-n} + (1-2n)F_{-n}^2 F_{-n+1} - 2)$.

(c) $\sum_{k=1}^{n} kF_{-k}^2 F_{-k+1} = \frac{1}{4}(-F_{-n+1}^3 - F_{-n}^3 + (1-2n)F_{-n+1}^2 F_{-n} + 2nF_{-n}^2 F_{-n+1} + 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.3. *For $n \geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} kL_{-k}^3 = \frac{1}{4}((2n-5)L_{-n+1}^3 + (6n-7)L_{-n}^3 - 3(2n-4)L_{-n+1}^2 L_{-n} - 3L_{-n}^2 L_{-n+1} + 49)$.

(b) $\sum_{k=1}^{n} kL_{-k+1}^2 L_{-k} = \frac{1}{4}(2(1-n)L_{-n+1}^3 + 2(2-n)L_{-n}^3 + (4n-7)L_{-n+1}^2 L_{-n} + (1-2n)L_{-n}^2 L_{-n+1} - 24)$.

(c) $\sum_{k=1}^{n} kL_{-k}^2 L_{-k+1} = \frac{1}{4}(-L_{-n+1}^3 - L_{-n}^3 + (1-2n)L_{-n+1}^2 L_{-n} + 2nL_{-n}^2 L_{-n+1} + 7)$.

Taking $r = 2, s = 1$ in Theorem 3.1 (a),(b) and (c), we obtain the following proposition.

Proposition 3.2. *If $r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k}^3 = \frac{1}{98}((14n-19)W_{-n+1}^3 + (112n-33)W_{-n}^3 - (63n-75)W_{-n+1}^2 W_{-n} + (21n-60)W_{-n}^2 W_{-n+1} + 19W_1^3 + 33W_0^3 - 75W_1^2 W_0 + 60W_0^2 W_1)$.

(b) $\sum_{k=1}^{n} kW_{-k+1}^2 W_{-k} = \frac{1}{98}(-(21n-4)W_{-n+1}^3 - (21n-25)W_{-n}^3 + (70n-39)W_{-n+1}^2 W_{-n} - (56n-41)W_{-n}^2 W_{-n+1} - 4W_1^3 - 25W_0^3 + 39W_1^2 W_0 - 41W_0^2 W_1)$.

(c) $\sum_{k=1}^{n} kW_{-k}^2 W_{-k+1} = \frac{1}{98}((7n-13)W_{-n+1}^3 + (7n-20)W_{-n}^3 + (41-56n)W_{-n+1}^2 W_{-n} + (84n-23)W_{-n}^2 W_{-n+1} + 13W_1^3 + 20W_0^3 - 41W_1^2 W_0 + 23W_0^2 W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 3.4. *For $n \geq 1$, Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} kP_{-k}^3 = \frac{1}{98}((14n-19)P_{-n+1}^3 + (112n-33)P_{-n}^3 - (63n-75)P_{-n+1}^2 P_{-n} + (21n-60)P_{-n}^2 P_{-n+1} + 19)$.

(b) $\sum_{k=1}^{n} kP_{-k+1}^2 P_{-k} = \frac{1}{98}(-(21n-4)P_{-n+1}^3 - (21n-25)P_{-n}^3 + (70n-39)P_{-n+1}^2 P_{-n} - (56n-41)P_{-n}^2 P_{-n+1} - 4)$.

(c) $\sum_{k=1}^{n} kP_{-k}^2 P_{-k+1} = \frac{1}{98}((7n-13)P_{-n+1}^3 + (7n-20)P_{-n}^3 + (41-56n)P_{-n+1}^2 P_{-n} + (84n-23)P_{-n}^2 P_{-n+1} + 13)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.5. *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} kQ_{-k}^3 = \frac{1}{98}((14n-19)Q_{-n+1}^3 + (112n-33)Q_{-n}^3 - (63n-75)Q_{-n+1}^2 Q_{-n} + (21n-60)Q_{-n}^2 Q_{-n+1} + 296)$.

(b) $\sum_{k=1}^{n} kQ_{-k+1}^2 Q_{-k} = \frac{1}{98}(-(21n-4)Q_{-n+1}^3 - (21n-25)Q_{-n}^3 + (70n-39)Q_{-n+1}^2 Q_{-n} - (56n-41)Q_{-n}^2 Q_{-n+1} - 248)$.

(c) $\sum_{k=1}^{n} kQ_{-k}^{2}Q_{-k+1} = \frac{1}{98}((7n-13)Q_{-n+1}^{3}+(7n-20)Q_{-n}^{3}+(41-56n)Q_{-n+1}^{2}Q_{-n}+(84n-23)Q_{-n}^{2}Q_{-n+1}+120)$.

Taking $r = 1, s = 2$ in Theorem 3.1 (a),(b) and (c), we obtain the following proposition.

Proposition 3.3. *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k}^{3} = \frac{1}{4900}(-(770n+1251)W_{-n+1}^{3} - (1260n+3848)W_{-n}^{3} - 12(70n+31)W_{-n}^{2}W_{-n+1} + 6(350n+575)W_{-n+1}^{2}W_{-n} + 1251W_{1}^{3} + 3848W_{0}^{3} - 3450W_{1}^{2}W_{0} + 372W_{0}^{2}W_{1})$.

(b) $\sum_{k=1}^{n} kW_{-k+1}^{2}W_{-k} = \frac{1}{4900}((350n+925)\,W_{-n+1}^{3} + 8\,(350n+575)\,W_{-n}^{3} - (1400n+3350)\,W_{-n+1}^{2}W_{-n} - 4\,(350n+225)\,W_{-n}^{2}W_{-n+1} - 925W_{1}^{3} - 4600W_{0}^{3} + 3350W_{1}^{2}W_{0} + 900W_{0}^{2}W_{1})$.

(c) $\sum_{k=1}^{n} kW_{-k}^{2}W_{-k+1} = \frac{1}{4900}(-(70n+101)W_{-n+1}^{3} - 8(70n+31)W_{-n}^{3} - 2(350n+225)W_{-n+1}^{2}W_{-n} + (1260n+628)W_{-n}^{2}W_{-n+1} + 101W_{1}^{3} + 248W_{0}^{3} + 450W_{1}^{2}W_{0} - 628W_{0}^{2}W_{1})$.

From the last proposition, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.6. *For $n \geq 1$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} kJ_{-k}^{3} = \frac{1}{4900}(-(770n+1251)J_{-n+1}^{3} - (1260n+3848)J_{-n}^{3} - 12(70n+31)J_{-n}^{2}J_{-n+1} + 6(350n+575)J_{-n+1}^{2}J_{-n} + 1251)$.

(b) $\sum_{k=1}^{n} kJ_{-k+1}^{2}J_{-k} = \frac{1}{4900}((350n+925)\,J_{-n+1}^{3}+8\,(350n+575)\,J_{-n}^{3}-(1400n+3350)\,J_{-n+1}^{2}J_{-n}-4\,(350n+225)\,J_{-n}^{2}J_{-n+1} - 925)$.

(c) $\sum_{k=1}^{n} kJ_{-k}^{2}J_{-k+1} = \frac{1}{4900}(-(70n+101)J_{-n+1}^{3} - 8(70n+31)J_{-n}^{3} - 2(350n+225)J_{-n+1}^{2}J_{-n} + (1260n+628)J_{-n}^{2}J_{-n+1} + 101)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.7. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kj_{-k}^{3} = \frac{1}{4900}(-(770n+1251)j_{-n+1}^{3}-(1260n+3848)j_{-n}^{3}-12(70n+31)j_{-n}^{2}j_{-n+1}+6(350n+575)j_{-n+1}^{2}j_{-n} + 26623)$.

(b) $\sum_{k=1}^{n} kj_{-k+1}^{2}j_{-k} = \frac{1}{4900}((350n+925)\,j_{-n+1}^{3}+8\,(350n+575)\,j_{-n}^{3}-(1400n+3350)\,j_{-n+1}^{2}j_{-n}-4\,(350n+225)\,j_{-n}^{2}j_{-n+1} - 27425)$.

(c) $\sum_{k=1}^{n} kj_{-k}^{2}j_{-k+1} = \frac{1}{4900}(-(70n+101)j_{-n+1}^{3} - 8(70n+31)j_{-n}^{3} - 2(350n+225)j_{-n+1}^{2}j_{-n} + (1260n+628)j_{-n}^{2}j_{-n+1} + 473)$.

4 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Horadam AF. Basic properties of a certain generalized sequence of numbers. Fibonacci Quarterly. 1965;3(3):161-176.

[2] Horadam AF. A generalized Fibonacci sequence. American Mathematical Monthly. 1961;68:455-459.

[3] Horadam AF. Special properties of the sequence $w_n(a, b; p, q)$. Fibonacci Quarterly. 1967;5(5):424-434.

[4] Horadam AF. Generating functions for powers of a certain generalized sequence of numbers. Duke Math. J. 1965;32:437-446.

[5] Sloane NJA. The on-line encyclopedia of integer sequences.
Available:http://oeis.org/

[6] Čerin Z. Formulae for sums of Jacobsthal–Lucas numbers. Int. Math. Forum. 2007;2(40):1969-1984.

[7] Čerin Z. Sums of squares and products of Jacobsthal numbers. Journal of Integer Sequences, 10, Article 07.2.5; 2007.

[8] Gnanam A, Anitha B. Sums of squares Jacobsthal numbers. IOSR Journal of Mathematics. 2015;11(6):62-64.

[9] Kiliç E, Taşçi D. The linear algebra of the Pell matrix. Boletín de la Sociedad Matemática Mexicana. 2005;3(11).

[10] Kılıc E. Sums of the squares of terms of sequence $\{u_n\}$. Proc. Indian Acad. Sci. (Math. Sci.). 2008;118(1):27-41.

[11] Soykan Y. Closed formulas for the sums of squares of generalized Fibonacci numbers. Asian Journal of Advanced Research and Reports. 2020;9(1):23-39.
Available:https://doi.org/10.9734/ajarr/2020/v9i130212

[12] Frontczak R. Sums of cubes over odd-index Fibonacci numbers. Integers. 2018;18.

[13] Soykan Y. Closed formulas for the sums of cubes of generalized Fibonacci numbers: Closed formulas of and $\sum_{k=0}^{n} W_k^3$ and $\sum_{k=1}^{n} W_{-k}^3$. Archives of Current Research International. 2020;20(2):58-69.
DOI: 10.9734/ACRI/2020/v20i230177

[14] Wamiliana, Suharsono, Kristanto PE. Counting the sum of cubes for Lucas and Gibonacci numbers, Science and Technology Indonesia. 2019;4(2):31-35.

[15] Chen L, Wang X. The power sums involving Fibonacci polynomials and their applications. Symmetry. 2019;11.
Doi.org/10.3390/sym11050635

[16] Frontczak R. Sums of powers of Fibonacci and Lucas numbers: A new bottom-up approach. Notes on Number Theory and Discrete Mathematics. 2018;24(2):94-103.

[17] Prodinger H. Sums of powers of Fibonacci polynomials. Proc. Indian Acad. Sci. (Math. Sci.). 2009;119(5):567-570.

[18] Raza Z, Riaz M, Ali MA. Some inequalities on the norms of special matrices with generalized Tribonacci and generalized Pell-Padovan sequences. arXiv; 2015.
Available:http://arxiv.org/abs/1407.1369v2

[19] Soykan Y. A closed formula for the sums of squares of generalized Tribonacci numbers. Journal of Progressive Research in Mathematics, Accepted.

[20] Prodinger H, Selkirk SJ. Sums of squares of Tetranacci numbers: A generating function approach; 2019.
Available:http://arxiv.org/abs/1906.08336v1

[21] Schumacher R. How to sum the squares of the Tetranacci numbers and the Fibonacci m-step numbers. Fibonacci Quarterly. 2019;57:168-175.

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