



LHAM Approach to Fractional Order Rosenau-Hyman and Burgers' Equations

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Authors contributions

This work was carried out in collaboration among all authors. SOA designed the study, ASO wrote the first draft of the manuscript, WNM edited the manuscript in the first instance. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i630192

Editor(s):

(1) Dr. Junjie Chen, University of Texas at Arlington, USA.

Reviewers:

(1) Mehmet Yavuz, University of Exeter, England.

(2) M. Abu-Shady, Menoufia University, Egypt.

Complete Peer review History: <http://www.sdiarticle4.com/review-history/56832>

Received: 02 March 2020

Accepted: 07 May 2020

Published: 14 May 2020

Original Research Article

Abstract

Fractional calculus has been found to be a great asset in finding fractional dimension in chaos theory, in viscoelasticity diffusion, in random optimal search etc. Various techniques have been proposed to solve differential equations of fractional order. In this paper, the Laplace-Homotopy Analysis Method (LHAM) is applied to obtain approximate analytic solutions of the nonlinear Rosenau-Hyman Korteweg-de Vries (KdV), $K(2, 2)$, and Burgers' equations of fractional order with initial conditions. The solutions of these equations are calculated in the form of convergent series. The solutions obtained converge to the exact solution when $\alpha = 1$, showing the reliability of LHAM.

Keywords: Laplace transform; Homotopy Analysis method; Laplace Homotopy Analysis method; Fractional derivative; KdV equation; Burger equation.

2010 Mathematics Subject Classification: 26A33, 34A08, 35A08, 44A10.

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1 Introduction

For a very long time, fractional calculus developed mainly as a field of mathematics that only pure mathematicians find useful but in recent decades, researchers have shown its usefulness and suitability in modeling and it has been used to model problems in biology [1], in fractional dimension in chaos theory, viscoelasticity diffusion [2], viscoplasticity [3], probability [4], wave propagation in porous media, in random optimal search, models of diffusion on fractal media, modeling oil pressure, rheology, electrical networks, and electromagnetic theory [5, 6, 7, 8]. Fractional differential equations has an advantage in modeling majorly because of its non-local property (it provides the possibility of different values of α in $\frac{d^\alpha}{dx^\alpha}$ forward or backward). Knowing that the next state of a non-Markov systems depends on both the present and the past states is why the fractional calculus very popular in science, technology and engineering [5].

Since there are no known methods of solutions for the exact solution of fractional differential equations, several methods have been proposed, developed and applied by different researchers. These methods include Adomian decomposition method [9, 10], Laplace transform method [11], variational iteration method, differential transform method [12], homotopy perturbation method [13], homotopy analysis method [14] etc. The homotopy analysis method (HAM) have been applied to so many nonlinear problems and it has proved very successful in many cases. The advantages of this method over other methods include the fact that many of the existing methods can be derived from it. It is important to also note that HAM possesses a setback in that it often requires the evaluation of some difficult quadratures. This setback is overcome when the method is used alongside with Laplace Transform and this gives birth to the Laplace Homotopy Analysis Method (LHAM). [15] presented an algorithm of the Laplace homotopy analysis method to obtain approximate solutions for linear and nonlinear oscillator fractional differential equations for any value of α ($1 < \alpha \leq 2$). With the LHAM, he constructed an analytic approximate solution for the linear harmonic fractional equation and fractional Van Der Pol oscillator equation and the results coincide with those of other methods. [16] applied LHAM to solve one-, two-, and three-dimensional fractional heat-like equations, and one-, two-, and three-dimensional wave-like equations subject to their respective initial conditions and then presented a procedure to construct the base function and gave a high order deformation equation in simple form. [17] used LHAM to find the exact solution to fractional biological population model with some given initial conditions. Mohamed et al. (2014) solved nonlinear time fractional gas dynamics equation and some fractional time-space derivatives nonlinear problem. [18] applied LHAM to systems of linear and nonlinear fractional differential equations and their results coincided perfectly with other methods used for the same problems. [19] applied the LHAM to get the solutions of a system of second-order boundary value problems and concluded that the method is simple and highly accurate. Integro-differential equation with initial conditions were solved using LHAM [20]. [21] applied LHAM to obtain approximate analytical solutions of the linear and nonlinear partial differential equations. Many recent researches have been carried out on the application and methods of solution of fractional derivatives in physical applications [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]

The fractional KdV type of equations is a nonlinear evolution equation with numerous applications in physical sciences and also in engineering fields [34, 35, 36]. [37] examined the KdV, the $K(2, 2)$, the Burgers and the cubic Boussinesq's equations by using the variation iteration method. Studies have been done on these partial differential equations but a detailed studies of their fractional order are just beginning as [36] are the only ones to have applied HAM for their solutions. To the best of our knowledge, LHAM has not been applied for the solution of fractional Rosenau-Hyman KdV, $K(2, 2)$ and Burgers' equations using Laplace homotopy analysis method (LHAM). In this paper, we develop the LHAM scheme for each of these equations and then apply the schemes to solve the problems, and the fast convergence of the method is shown.

2 Preliminary Definitions and Theorems

Some important definitions and theorems are stated in this section as preliminaries to main work.

Definition 2.1. The gamma function is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \text{Re}(z) > 0, \quad z \in \mathbb{Z}. \quad (2.1)$$

Replacing z with $z + 1$ and then integrating by part gives

$$\Gamma(z + 1) = \int_0^{\infty} e^{-t} t^z dt = -e^{-t} t^z \Big|_{t=0}^{t=\infty} + z \int_0^{\infty} e^{-t} t^{z-1} dt = z\Gamma(z), \quad (2.2)$$

and as a consequence,

$$\Gamma(n + 1) = n!, \quad \Gamma(1/2) = \sqrt{\pi}.$$

Definition 2.2. Laplace transform of a function $f(t)$ is defined as the improper integral

$$F(s) = \mathcal{L}\{f(t); s\} = \int_0^{\infty} e^{-st} f(t) dt, \quad (2.3)$$

such that the integral converges and exist.

Definition 2.3. The differential operator is defined as

$$D_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha = 0. \end{cases}$$

The most frequently used definitions for D_t^α , $\alpha \in \mathbb{R}$ are those given by Grünwald-Letnikov, Riemann-Liouville and Caputo (Miller and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999; Petráš, 2011).

Definition 2.4. [Podlubny, 1999] The Riemann-Liouville derivative of fractional order $\alpha > 0$ is

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau.$$

The Riemann-Liouville fractional integral of $f(x, t)$ with respect to t of order α is defined as

$$I^\alpha f(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(x, \tau) d\tau. \quad t, \alpha > 0.$$

where Γ is the Gamma function.

Definition 2.5. [Podlubny, 1999] Suppose that $\alpha > 0$, $t > 0$, $\alpha, t \in \mathbb{R}$, the Caputo fractional derivative of order α is defined as

$$D_t^\alpha f(x, t) = I^{n-\alpha} D^n f(x, t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t) & \alpha = n \in \mathbb{N} \end{cases}, \quad (2.4)$$

The Caputo fractional derivative and Riemann-Liouville integral operator are related by the relations [Özpinar, 2018]

$$D^\alpha (I^\alpha f(x, t)) = f(x, t) \text{ and}$$

$$I^\alpha (D^\alpha f(x, t)) = I^\alpha (I^{n-\alpha} f^{(n)}(x, t)) = I^n f^{(n)}(x, t) = f(x, t) - \sum_{k=0}^{n-1} f^{(k)}(x, 0) \frac{t^k}{k!}.$$

Lemma 2.1. Suppose $p > 0$ and $F(s)$ is the Laplace transform of $f(t)$. The Laplace transform of the Riemann-Liouville fractional derivative is given by

$$\begin{aligned} \mathcal{L}\{D_t^\alpha f(t); s\} &= s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0}, \\ &= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{n-k-1} [D^k J^{n-\alpha} f(t)]_{t=0}, \quad n-1 < \alpha < n. \end{aligned}$$

Lemma 2.2. [Zurigat, 2011] Suppose $p > 0$ and $F(s)$ is the Laplace transform of $f(t)$. The Laplace transform of the Caputo fractional derivative of order α is given by

$$\mathcal{L}\{D_t^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha < n. \quad (2.5)$$

3 Fractional Rosenau-Hyman and Burgers' Equations

A generalised Korteweg-de Vries (KdV) equation with $m, n > 1$ is the equation [38]

$$u_t + (u^m)_x + (u^n)_{xxx} = 0. \quad (3.1)$$

The Rosenau-Hyman equation, named after the Rosenau and Hyman based on their study of solitons, is a Korteweg-de Vries-like (KdV) equations having compacton solutions. The equation is the special case of compactons with $m = n$ in Eq. (3.1) so that it is written as

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0. \quad (3.2)$$

The Burgers' equation appears in fluid mechanics and it is a modified KdV (mKdV) equation ($m = 2, n = 1$) [37, 39]. It is represented by the equation

$$u_t + \frac{1}{2}(u^2)_x - u_{xx} = 0. \quad (3.3)$$

By replacing the integer-order time derivatives with fractional derivatives, the fractional Rosenau-Hyman KdV and $K(2, 2)$ are given as

$$D_t^\alpha u + a(u^2)_x + u_{xxx} = 0, \quad (3.4)$$

$$D_t^\alpha u + a(u^2)_x + (u^2)_{xxx} = 0, \quad (3.5)$$

respectively, where $a \neq 0, t > 0$ and $0 < \alpha \leq 1$ [36, 37]. Replacing the integer time order derivatives in Eq.(3.3) by fractional derivatives we have the fractional Burgers' equation as

$$D_t^\alpha u + \frac{1}{2}(u^2)_x - u_{xx} = 0, \quad t > 0, x \in \mathbb{R}, 0 < \alpha \leq 1.$$

4 Methodology

4.1 Homotopy analysis method

Consider a fractional partial differential equation of the form

$$\mathcal{N}(u(x, t)) = 0,$$

where \mathcal{N} is a nonlinear partial fractional differential equation, x and t are independent variables and $u(x, t)$ is an unknown function. Construct the zero-order deformation as follows

$$(1 - q)\mathcal{L}^*[v(x, t; q) - u_0(x, t)] = q\hbar\mathcal{N}[v(x, t; q)], \quad (4.1)$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, \mathcal{L}^* is a linear operator and $u_0(x, t)$ is an initial guess of $u(x, t)$, $v(x, t; q)$ is an unknown function of independent variables x, t, q . When $q = 0$ and $q = 1$, it holds that $v(x, t; 0) = u_0(x, t)$, $v(x, t; 1) = u(x, t)$ respectively. Thus as q increases from 0 to 1, $v(x, t; q)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$. Taylor's expansion of $v(x, t; q)$ with respect to q gives

$$v(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m, \quad (4.2)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m v(x, t; q)}{\partial q^m} \right|_{q=0}.$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter \hbar are properly chosen, then Eq.(4.2) converges at $q = 1$. Hence we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t),$$

Define a vector $\vec{u}_n(x, t) = \{u_0(x, t), \dots, u_n(x, t)\}$, differentiate (4.1) m times with respect to the embedding parameter q , set $q = 0$ and dividing through by $m!$ so that the m th-order deformation of (4.1) is given by

$$\mathcal{L}^*[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathcal{R}(\vec{u}_{m-1}(x, t)), \quad (4.3)$$

where

$$\mathcal{R}(\vec{u}_{m-1}(x, t)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}(v(x, t; q))}{\partial q^{m-1}} \right|_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1 \\ 1, & \text{if } m > 1 \end{cases}.$$

4.2 Laplace homotopy analysis method

Consider the fractional differential equation

$$D_t^\alpha u(t) = g(u(t), u_x(t), u_{xx}(t)), \quad t \geq 0, \quad 0 < \alpha \leq 1. \quad (4.4)$$

with the initial conditions:

$$u(0) = a, \quad (4.5)$$

where D_t^α is Caputo's derivative. Apply the Laplace transform to both sides of Eq.(4.4) and by the linearity of Laplace transforms we get

$$\mathcal{L}(D_t^\alpha u(t)) = \mathcal{L}(g(u(t), u_x(t), u_{xx}(t))).$$

Using the initial condition (4.5), then we get

$$s^\alpha \tilde{u}(s) - s^{\alpha-1} a = \mathcal{L}(g(u(t), u_x(t), u_{xx}(t))),$$

and consequently,

$$\tilde{u}(s) = \frac{a}{s} + \frac{1}{s^\alpha} \mathcal{L}(g(u(t), u_x(t), u_{xx}(t))), \quad (4.6)$$

where $\mathcal{L}(u(t)) = \tilde{u}(s)$. The zero-order deformation equation of the Laplace equation (4.6) has the form

$$(1-q)[\tilde{\phi}(s; q) - \tilde{u}_0(s)] = q\hbar \left[\tilde{\phi}(s; q) - \frac{a}{s} - \frac{1}{s^\alpha} \mathcal{L}(g(u(t), u_x(t), u_{xx}(t))) \right], \quad (4.7)$$

where $q \in [0, 1]$ is an embedding parameter. When $q = 0$ and $q = 1$, we have

$$\tilde{\phi}(s; 0) = \tilde{u}_0(s) \quad \text{and} \quad \tilde{\phi}(s; 1) = \tilde{u}(s)$$

respectively. Thus, as q increases from 0 to 1, $\tilde{\phi}(s; q)$ varies from $\tilde{u}_0(s)$ to $\tilde{u}(s)$ and the Taylor's expansion of $\tilde{\phi}(s; q)$

$$\tilde{\phi}(s; q) = \tilde{u}_0(s) + \sum_{m=1}^{\infty} \tilde{u}_m(s) q^m, \quad (4.8)$$

where

$$\tilde{u}_m(s) = \frac{1}{m!} \left. \frac{\partial^m \tilde{\phi}(s; q)}{\partial q^m} \right|_{q=0}.$$

If the auxiliary parameter \hbar and the initial guesses $\tilde{u}_0(s)$ are properly chosen, then (4.8) converges at $q = 1$ and we have

$$\tilde{u}(s) = \tilde{u}_0(s) + \sum_{m=1}^{\infty} \tilde{u}_m(s).$$

Define the vector

$$\vec{\tilde{u}}_m(s) = \{\tilde{u}_0(s), \tilde{u}_1(s), \dots, \tilde{u}_m(s)\}.$$

Differentiating Eq.(4.7) m times with respect to q , setting $q = 0$, $\hbar = -1$ and finally dividing through by $m!$, we have the m th-order deformation equation

$$\tilde{u}_m(s) = \chi_m \tilde{u}_{m-1}(s) - R_m \left(\vec{\tilde{u}}_{m-1}(s) \right), \quad (4.9)$$

where

$$R_m \left(\vec{\tilde{u}}_{m-1}(s) \right) = \tilde{u}_{m-1}(s) - \frac{a}{s} (1 - \chi_m) - \frac{1}{s^\alpha} \left(\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[\mathcal{L}(g(t, \phi(t; q), \frac{d}{dt} \phi(t, q))) \right]_{q=0} \right), \quad (4.10)$$

and

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1 \\ 1, & \text{if } m > 1 \end{cases}.$$

Taking the inverse Laplace transform of (4.9) gives a power series solution

$$u(t) = \sum_{i=0}^{\infty} u_i(t). \quad (4.11)$$

5 Implementation of LHAM for Rosenau-Hyman KdV, $K(2, 2)$ and Burgers' Equations of Fractional Order

5.1 LHAM solution of fractional Rosenau-Hyman KdV equation

The fractional Rosenau-Hyman KdV equation

$$D_t^\alpha u(x, t) + a(u^2)_x(x, t) + u_{xxx}(x, t) = g(x, t), \quad (5.1)$$

where a is a constant, $0 < \alpha \leq 1$ and $g(x, t)$ is a function of x and t with the initial condition

$$u(x, 0) = f(x). \quad (5.2)$$

Applying Laplace transform to Eq.(5.1), dividing through by s^α , substituting Eq.(5.2) and rearranging gives

$$u(x, s) - \frac{f(x)}{s} + \frac{1}{s^\alpha} \mathcal{L} (a(u^2)_x(x, t) + u_{xxx}(x, t) - g(x, t)) = 0. \quad (5.3)$$

The zeroth-order deformation is

$$(1 - q) [v(x, s; q) - u_0(x, s)] = q\hbar \mathcal{N}v(x, s; q), \quad (5.4)$$

where

$$\mathcal{N}v(x, s; q) = \left[v(x, s; q) - \frac{f(x)}{s} + \frac{1}{s^\alpha} \mathcal{L} (a(u^2)_x(x, t) + u_{xxx}(x, t) - g(x, t)) \right],$$

$q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ an auxiliary parameter, and $u_0(x, s)$ is an initial guess of $u(x, s)$, $v(x, s; q)$ is an unknown function and as q increases from 0 to 1, $v(x, s; q)$ varies from the initial guess $u_0(x, s)$ to the solution $u(x, s)$ i.e.

$$v(x, s; 0) = u_0(x, s), \quad v(x, s; 1) = u(x, s).$$

The Taylor's expansion of $v(x, s; q)$ is

$$v(x, s; q) = u_0(x, s) + \sum_{m=1}^{\infty} u_m(x, s)q^m, \quad (5.5)$$

where

$$u_m(x, s) = \frac{1}{m!} \left. \frac{\partial^m v(x, s; q)}{\partial q^m} \right|_{q=0}.$$

For a proper choice of the auxiliary linear operator, initial guess, and auxiliary parameter \hbar , Eq.(5.5) converges at $q = 1$. Hence we have

$$u(x, s) = u_0(x, s) + \sum_{m=1}^{\infty} u_m(x, s),$$

and define a vector $\vec{u}_n(x, s) = \{u_0(x, s), \dots, u_n(x, s)\}$, differentiate Eq.(5.4) m -times with respect to q , set $q = 0$, $\hbar = -1$ and finally dividing through by $m!$ to obtain the m th-order deformation equation

$$u_m(x, s) = \chi_m u_{m-1}(x, s) + \left(\frac{\mathcal{L}}{s^\alpha} g(x, t) + \frac{f(x)}{s} \right) (1 - \chi_m) - \left[u_{m-1}(x, s) + \frac{\mathcal{L}}{s^\alpha} \left(a \sum_{i=0}^{m-1} (u_i u_{m-1-i})_x(x, t) + (u_{m-1})_{xxx}(x, t) \right) \right], \quad (5.6)$$

where

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1 \\ 1, & \text{if } m > 1 \end{cases}.$$

Applying the inverse Laplace transform to both sides of Eq.(5.6) then we have a power series solution

$$u(x, t) = \sum_{i=0}^{\infty} u_i(x, t).$$

Now consider

$$D_t^\alpha u(x, t) - 3(u^2)_x + (u)_{xxx} = 0, \quad 0 < \alpha \leq 1, \quad u(x, 0) = 6x, \quad (5.7)$$

where we have $a = -3$, $g(x, t) = 0$, $u(x, 0) = f(x) = 6x$. In this case Eq.(5.6) becomes

$$u_m(x, s) = \chi_m u_{m-1}(x, s) + \frac{6x}{s} (1 - \chi_m) - u_{m-1}(x, s) - \frac{1}{s^\alpha} \mathcal{L} \left(-3 \sum_{i=0}^{m-1} (u_i u_{m-1-i})_x(x, t) + (u_{m-1})_{xxx}(x, t) \right),$$

with

$$u_0(x, s) = \frac{6x}{s}.$$

Thus,

$$u_1(x, s) = \frac{6^3 x}{s^{\alpha+1}}, \quad u_2(x, s) = \frac{2 \times 6^5 x}{s^{2\alpha+1}}, \dots,$$

and so

$$u(x, s) = \frac{6x}{s} + \frac{6^3 x}{s^{\alpha+1}} + \frac{2 \times 6^5 x}{s^{2\alpha+1}}$$

Taking the inverse Laplace transform, we get

$$u(x, t) = 6x + \frac{6^3 x}{\Gamma(\alpha + 1)} t^\alpha + \frac{2 \times 6^5 x}{\Gamma(2\alpha + 1)} t^{2\alpha} + \quad (5.8)$$

As a test case, set $\alpha = 1$, and we have

$$u(x, t) = 6x (1 + 36t + 36^2 t^2 + 36^3 t^3 + \dots), \quad |36t| < 1,$$

which is the exact solution when $\alpha = 1$. The solution for the fractional order problem is obtained as

$$\begin{aligned} u(x, t) &= 6x + \frac{6^3 x}{\Gamma\left(\frac{19}{10}\right)} t^{\frac{9}{10}} + \frac{2 \times 6^5 x}{\Gamma\left(\frac{28}{10}\right)} t^{\frac{18}{10}} + \dots, \quad \alpha = 0.9. \\ u(x, t) &= 6x + \frac{6^3 x}{\Gamma\left(\frac{18}{10}\right)} t^{\frac{8}{10}} + \frac{2 \times 6^5 x}{\Gamma\left(\frac{26}{10}\right)} t^{\frac{16}{10}} + \dots, \quad \alpha = 0.8. \\ u(x, t) &= 6x + \frac{6^3 x}{\Gamma\left(\frac{17}{10}\right)} t^{\frac{7}{10}} + \frac{2 \times 6^5 x}{\Gamma\left(\frac{24}{10}\right)} t^{\frac{14}{10}} + \dots, \quad \alpha = 0.7. \\ u(x, t) &= 6x + \frac{6^3 x}{\Gamma\left(\frac{16}{10}\right)} t^{\frac{6}{10}} + \frac{2 \times 6^5 x}{\Gamma\left(\frac{22}{10}\right)} t^{\frac{12}{10}} + \dots, \quad \alpha = 0.6. \end{aligned}$$

5.2 LHAM solution of fractional Rosenau-Hyman $K(2, 2)$ equation

The $K(2, 2)$ equation is given as

$$D_t^\alpha u + (u^2)_x + (u^2)_{xxx} = 0, \quad u(x, 0) = x, \quad 0 < \alpha \leq 1. \quad (5.9)$$

Taking the Laplace transform, we have

$$s^\alpha u(x, s) - s^{\alpha-1} u(x, 0) + \mathcal{L} \left((u^2)_x(x, t) + (u^2)_{xxx}(x, t) \right) = 0,$$

and dividing through by s^α , we obtain

$$u(x, s) - \frac{x}{s} + \frac{1}{s^\alpha} \mathcal{L} \left((u^2)_x(x, t) + (u^2)_{xxx}(x, t) \right) = 0.$$

The zeroth-order deformation equation is

$$(1 - q) [v(x, s; q) - u_0(x, s)] = q\hbar \left[v(x, s; q) - \frac{x}{s} + \frac{1}{s^\alpha} \mathcal{L} \left((u^2)_x(x, t) + (u^2)_{xxx}(x, t) \right) \right], \quad (5.10)$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ an auxiliary parameter, and $u_0(x, s)$ is an initial guess of $u(x, s)$, $v(x, s; q)$ is an unknown function with the condition that

$$v(x, s; 0) = u_0(x, s), \quad v(x, s; 1) = u(x, s),$$

and as q increases from 0 to 1, $v(x, s; q)$ varies from the initial guess $u_0(x, s)$ to the solution $u(x, s)$. Taylor's expansion of $v(x, s; q)$ is

$$v(x, s; q) = u_0(x, s) + \sum_{m=1}^{\infty} u_m(x, s)q^m, \quad (5.11)$$

where

$$u_m(x, s) = \frac{1}{m!} \left. \frac{\partial^m v(x, s; q)}{\partial q^m} \right|_{q=0}.$$

For a proper choice of the auxiliary linear operator, the initial guess, and the auxiliary parameter \hbar , Eq.(5.11) converges at $q = 1$ so that

$$u(x, s) = u_0(x, s) + \sum_{m=1}^{\infty} u_m(x, s).$$

Define a vector

$$\vec{u}_n(x, s) = \{u_0(x, s), \dots, u_n(x, s)\}.$$

Differentiate Eq.(5.10) m times with respect to q , set $q = 0$, $\hbar = -1$ and finally dividing by $m!$, we get the m th-order deformation equation

$$u_m(x, s) = \chi_m u_{m-1}(x, s) - u_{m-1}(x, s) + \frac{x}{s} (1 - \chi_m) - \frac{\mathcal{L}}{s^\alpha} \left(\sum_{i=0}^{m-1} (u_i u_{m-1-i})_x(x, t) + \sum_{i=0}^{m-1} (u_i u_{m-1-i})_{xx}(x, t) \right), \quad (5.12)$$

where

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1 \\ 1, & \text{if } m > 1 \end{cases},$$

and

$$u_0(x, s) = \frac{x}{s}.$$

From Eq.(5.10), we have

$$u_1(x, s) = -\frac{2x}{s^{\alpha+1}}, \quad u_1(x, s) = \frac{-2x}{s^\alpha}, \quad u_2(x, s) = \frac{8x}{s^{2\alpha+1}}, \quad u_3(x, s) = -\frac{48x}{s^{3\alpha+1}}.$$

Thus

$$u(x, s) = \frac{x}{s} - \frac{2x}{s^{\alpha+1}} + \frac{8x}{s^{2\alpha+1}} - \dots,$$

and taking the inverse Laplace transform, we get

$$u(x, t) = x - \frac{2x}{\Gamma(\alpha+1)} t^\alpha + \frac{8x}{\Gamma(2\alpha+1)} t^{2\alpha} - + \dots. \quad (5.13)$$

As a test case, set $\alpha = 1$, and we have

$$u(x, t) = x \left[\lim_{n \rightarrow +\infty} \sum_{m=0}^n (-1)^m (2t)^m \right] = \frac{x}{1+2t}, \quad |2t| < 1.$$

and for fractional values of α ,

$$\begin{aligned} u(x, t) &= x - \frac{2x}{\Gamma\left(\frac{19}{10}\right)} t^{\frac{9}{10}} + \frac{8x}{\Gamma\left(\frac{28}{10}\right)} t^{\frac{18}{10}} - + \dots, \alpha = 0.9. \\ u(x, t) &= x - \frac{2x}{\Gamma\left(\frac{18}{10}\right)} t^{\frac{8}{10}} + \frac{8x}{\Gamma\left(\frac{26}{10}\right)} t^{\frac{16}{10}} - + \dots, \alpha = 0.8. \\ u(x, t) &= x - \frac{2x}{\Gamma\left(\frac{17}{10}\right)} t^{\frac{7}{10}} + \frac{8x}{\Gamma\left(\frac{24}{10}\right)} t^{\frac{14}{10}} - + \dots, \alpha = 0.7. \\ u(x, t) &= x - \frac{2x}{\Gamma\left(\frac{16}{10}\right)} t^{\frac{6}{10}} + \frac{8x}{\Gamma\left(\frac{22}{10}\right)} t^{\frac{12}{10}} - + \dots, \alpha = 0.6. \\ u(x, t) &= x - \frac{2x}{\Gamma\left(\frac{15}{10}\right)} t^{\frac{5}{10}} + \frac{8x}{\Gamma\left(\frac{20}{10}\right)} t^{\frac{10}{10}} - \frac{48x}{\Gamma\left(\frac{25}{10}\right)} t^{\frac{15}{10}} + \dots, \alpha = 0.5. \end{aligned}$$

5.3 LHAM solution of fractional Burgers' equation

The fractional Burgers' equation is the modified KdV (mKdV) given as

$$D_t^\alpha u + \frac{1}{2} (u^2)_x - (u)_{xx} = 0, \quad u(x, 0) = x, \quad 0 < \alpha \leq 1. \quad (5.14)$$

Taking the Laplace transform of Eq.(5.14) and rearranging we have

$$u(x, s) - \frac{x}{s} + \frac{\mathcal{L}}{s^\alpha} \left(\frac{1}{2} (u^2)_x(x, t) - (u)_{xx}(x, t) \right) = 0, \quad u(x, 0) = x.$$

The zero-order deformation equation is given by

$$(1 - q) [v(x, s; q) - u_0(x, s)] = q\hbar \left[v(x, s; q) - \frac{x}{s} + \frac{1}{s^\alpha} \mathcal{L} \left(\frac{1}{2} (u^2)_x(x, t) - (u)_{xx}(x, t) \right) \right], \quad (5.15)$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, and $u_0(x, s)$ is an initial guess of $u(x, s)$, $v(x, s; q)$ is an unknown function. It follows that

$$v(x, s; 0) = u_0(x, s), \quad v(x, s; 1) = u(x, s).$$

Thus as q increases from 0 to 1, $v(x, s; q)$ varies from the initial guess $u_0(x, s)$ to the solution $u(x, s)$. Taylor's expansion of $v(x, s; q)$ is

$$v(x, s; q) = u_0(x, s) + \sum_{m=1}^{\infty} u_m(x, s) q^m, \quad (5.16)$$

where

$$u_m(x, s) = \frac{1}{m!} \left. \frac{\partial^m v(x, s; q)}{\partial q^m} \right|_{q=0}.$$

Properly choosing the auxiliary linear operator, the initial guess, and the auxiliary parameter \hbar , then Eq.(5.16) converges at $q = 1$ and we therefore have

$$u(x, s) = u_0(x, s) + \sum_{m=1}^{\infty} u_m(x, s).$$

Define a vector $\vec{u}_n(x, s) = \{u_0(x, s), \dots, u_n(x, s)\}$. Differentiating Eq.(5.15) m times with respect to q , setting $q = 0$, and finally dividing by $m!$, we have the m th-order deformation equation as

$$u_m(x, s) = \chi_m u_{m-1}(x, s) - u_{m-1}(x, s) + \frac{x}{s} (1 - \chi_m) - \frac{\mathcal{L}}{s^\alpha} \left(\frac{1}{2} \sum_{i=0}^{m-1} (u_i u_{m-1-i})_x(x, t) - (u_{m-1})_{xx}(x, t) \right), \quad (5.17)$$

and

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1 \\ 1, & \text{if } m > 1 \end{cases}.$$

The inverse Laplace transform of Eq.(5.17) gives

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t),$$

where

$$u_0(x, s) = \frac{x}{s}.$$

Thus we have

$$u_1(x, s) = -\frac{x}{s^{\alpha+1}}, \quad u_2(x, s) = \frac{2x}{s^{2\alpha+1}}, \quad u_3(x, s) = -\frac{6x}{s^{3\alpha+1}}, \dots,$$

and so

$$u(x, s) = \frac{x}{s} - \frac{x}{s^{\alpha+1}} + \frac{2x}{s^{2\alpha+1}} - \dots,$$

Taking the inverse Laplace transform, we have

$$u(x, t) = x - \frac{x}{\Gamma(\alpha + 1)} t^\alpha + \frac{2x}{\Gamma(2\alpha + 1)} t^{2\alpha} - \dots \quad (5.18)$$

For a test case, let $\alpha = 1$, then we have

$$u(x, t) = x \left(\lim_{n \rightarrow +\infty} \sum_{m=0}^n (-1)^m t^m \right) = \frac{x}{1+t},$$

which coincides with the exact solution. Thus for fractional α , we have

$$u(x, t) = x - \frac{x}{\Gamma\left(\frac{19}{10}\right)} t^{\frac{9}{10}} + \frac{2x}{\Gamma\left(\frac{28}{10}\right)} t^{\frac{18}{10}} - + \dots \quad \alpha = 0.9.$$

$$u(x, t) = x - \frac{x}{\Gamma\left(\frac{18}{10}\right)} t^{\frac{8}{10}} + \frac{2x}{\Gamma\left(\frac{26}{10}\right)} t^{\frac{16}{10}} - + \dots \quad \alpha = 0.8.$$

$$u(x, t) = x - \frac{x}{\Gamma\left(\frac{17}{10}\right)} t^{\frac{7}{10}} + \frac{2x}{\Gamma\left(\frac{24}{10}\right)} t^{\frac{14}{10}} - + \dots \quad \alpha = 0.7.$$

$$u(x, t) = x - \frac{x}{\Gamma\left(\frac{16}{10}\right)} t^{\frac{6}{10}} + \frac{2x}{\Gamma\left(\frac{22}{10}\right)} t^{\frac{12}{10}} - + \dots \quad \alpha = 0.6.$$

6 Discussion of Results

Analytic approximate solutions of Rosenau-Hyman KdV, $K(2, 2)$ and Burgers' equations of fractional order were obtained using Laplace Homotopy Analysis method. Results obtained are tested for the analog integer order and the solutions are found to converge to the exact solution. Consequently, the results obtained here can be extended to the fractional order.

7 Conclusion

This work has verified the suitability of the Laplace homotopy analysis method for obtaining the analytic approximate solutions of the fractional order Rosenau-Hyman KdV, $K(2, 2)$ and Burgers' equations. The results obtained in this paper can be extended other equations in this family of equations. It will be useful to find out if this method will be able to examine these equations for their soliton (i.e. traveling wave) solutions.

Competing Interests

Authors have declared that no competing interests exist.

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