Open Journal of Mathematical Sciences

PSR Press

Article

A method to compute the determinant of square matrices of order five and six

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Received: 13 September 2018; Accepted: 11 March 2019; Published: 1 April 2019.

Abstract: In this paper we present a new method to compute the determinants of square matrices of order 5 and 6. To prove the main results we have combined the Farhadian's Duplex Fraction method and Salihu's method to reduce the order of determinants to second order. Hence, this paper gives the possibility to develop a general method to compute the determinants of higher order.

Keywords: Determinants, Farhadian's Duplex fraction, twice Dodgson's condensation.

MSC: 15A15, 11C20, 65F40.

1. Introduction and main definitions

Let A be a $n \times n$ matrix:

$$[A_{n\,\times\,n}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Definition 1. The determinant of the matrix of order $n\times n$ is the sum

$$|A_{n\times n}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{S_n} \varepsilon_{j_1 j_2 \ldots j_n} \cdot a_{j_1} \cdot a_{j_2} \cdot \ldots \cdot a_{j_n},$$

ranging over the symmetric permutation group S_n, where

$$\varepsilon_{j_1 j_2 \ldots j_n} = \begin{cases} +1, & \text{if } j_1 j_2 \ldots j_n, \text{ is an even permutation} \\ -1, & \text{if } j_1 j_2 \ldots j_n, \text{ is an odd permutation.} \end{cases}$$

Definition 2. [1] Let $A_2 = \begin{bmatrix} a_{11} & a_{22} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$ and $B_2 = \begin{bmatrix} b_{11} & b_{22} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$ are two real matrices of order 2×2. If $|B_2| \neq 0$ and $b_{ij} \neq 0$, $(\forall i, j = 1, 2)$, then the duplex fraction or duplex division of the determinant of $|A_2|$ on $|B_2|$ is defined as follows

$$\frac{|A_2|}{\overline{|B_2|}} = \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{\overline{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}}} = \frac{\begin{vmatrix} \frac{a_{11}}{b_{11}} & \frac{a_{12}}{b_{12}} \\ \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}}.$$

Definition 3. [2] Let $A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ and $B_3 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times 3}$ are two matrices of order 3×3 such that $b_{ij} \neq 0$, $(\forall i, j = 1, 2, 3)$ and B_3 is doubly nonsingular, then the star fraction of A_3 on B_3 is defined as

$$\left(\frac{A_3}{B_3}\right)^* = \left(\frac{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times 3}}\right) = \frac{\begin{vmatrix} \frac{\begin{vmatrix} \frac{a_{11}}{b_{11}} & \frac{a_{12}}{b_{12}} \\ \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}} & \frac{\begin{vmatrix} \frac{a_{12}}{b_{12}} & \frac{a_{13}}{b_{13}} \\ \frac{a_{22}}{b_{22}} & \frac{a_{23}}{b_{23}} \end{vmatrix}}{\begin{vmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \end{vmatrix}} \\ \frac{\begin{vmatrix} \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} \\ \frac{a_{31}}{b_{31}} & \frac{a_{32}}{b_{32}} \end{vmatrix}}{\begin{vmatrix} b_{21} & b_{122} \\ b_{31} & b_{32} \end{vmatrix}} & \frac{\begin{vmatrix} \frac{a_{22}}{b_{22}} & \frac{a_{23}}{b_{23}} \\ \frac{a_{32}}{b_{32}} & \frac{a_{33}}{b_{33}} \end{vmatrix}}{\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}} \end{vmatrix}^*}{\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}}.$$

Definition 4. [3] Let $B_n = [b_{ij}]_{n\times n}$ be a square real matrix of order n, then the Dodgson's condensation of matrix B_n is a $(n-1) \times (n-1)$ matrix defined as:

$$DC(B_n) = \begin{bmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} & \cdots & \begin{vmatrix} b_{1(n-1)} & b_{1n} \\ b_{2(n-1)} & b_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} b_{(n-1)1} & b_{(n-1)2} \\ b_{n1} & b_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} b_{(n-1)(n-1)} & b_{(n-1)n} \\ b_{n(n-1)} & b_{nn} \end{vmatrix} \end{bmatrix}_{(n-1)\times(n-1)}.$$

Definition 5. Let $A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times 4}$ be a square matrix of order 4, then the twice Dodgsons's condensation is defined as

$$DC(A_4) = \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{bmatrix}_{3\times 3},$$

and
$$DC(DC(A_4)) = \begin{bmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{bmatrix}_{2\times 2}.$$

The twice Dodgson's condensation Duplex fraction for the square matrix of order 4 is defined as follows:

$$FDC\left(DC\left(A_4\right)\right)=\begin{bmatrix} \dfrac{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix}}{a_{22}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix}}{a_{23}} \\ \dfrac{\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix}}{a_{32}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix}}{a_{33}} \end{bmatrix}.$$

Definition 6. Let $A_5=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}_{5\times 5}$ be a square matrix of order 5, then the thrice Dodgsons's condensation is defined as follows:

$$DC\left(A_5\right)=\begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{bmatrix}_{4\times 4},$$

and, $FDC\left(DC\left(A_5\right)\right)=$

$$\begin{bmatrix} \dfrac{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix}}{a_{22}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix}}{a_{23}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix}}{a_{24}} \\ \dfrac{\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix}}{a_{32}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix}}{a_{33}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix}}{a_{34}} \\ \dfrac{\begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix}}{a_{42}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix}}{a_{43}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix}}{a_{44}} \end{bmatrix}_{3\times 3},$$

The thrice Dodgson's condensation Duplex fraction for the square matrix of order 5 is defined as follows: $FDC\left(DC\left(DC\left(A_5\right)\right)\right) =$

$$\begin{bmatrix}
\frac{\begin{vmatrix}
\frac{\begin{vmatrix}\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}&\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}\\\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\end{vmatrix}}{a_{22}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}&\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}\\\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\end{vmatrix}}{a_{23}} \\
\frac{\begin{vmatrix}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\\\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\end{vmatrix}}{a_{32}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix}}{a_{33}}
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\frac{\begin{vmatrix}\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}&\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}\\\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\end{vmatrix}}{a_{23}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}&\begin{vmatrix}a_{14}&a_{15}\\a_{24}&a_{25}\end{vmatrix}\\\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\end{vmatrix}}{a_{24}} \\
\frac{\begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix}}{a_{33}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\\\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\end{vmatrix}}{a_{34}}
\end{vmatrix}}{\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}} \\
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\frac{\begin{vmatrix}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\\\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\end{vmatrix}}{a_{32}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix}}{a_{33}} \\
\frac{\begin{vmatrix}\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\\\begin{vmatrix}a_{41}&a_{42}\\a_{51}&a_{52}\end{vmatrix}&\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}\end{vmatrix}}{a_{42}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\\\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}&\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}\end{vmatrix}}{a_{43}}
\end{vmatrix}}{\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}} &
\frac{\begin{vmatrix}
\frac{\begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix}}{a_{33}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\\\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\end{vmatrix}}{a_{34}} \\
\frac{\begin{vmatrix}\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\\\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}&\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}\end{vmatrix}}{a_{43}} &
\frac{\begin{vmatrix}\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\\\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}&\begin{vmatrix}a_{44}&a_{45}\\a_{54}&a_{55}\end{vmatrix}\end{vmatrix}}{a_{44}}
\end{vmatrix}}{\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}}
\end{bmatrix}_{2\times 2}$$

2. Some useful Lemmas

To prove our main results we need the following lemmas.

Lemma 7. *(Salihu's method [4]) The determinant of the square matrix* $A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ *is equal to:*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \frac{\begin{vmatrix} \begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} \end{vmatrix} & \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} \\ \begin{vmatrix} a_{21} & \cdots & a_{2(n-1)} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} \end{vmatrix} & \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} \end{vmatrix}}.$$

Lemma 8. *[1] Given a square matrix*

$$A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times 4}$$

of order 4 such that

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} > 0 \text{ and } \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0.$$

Then

$$det\left(A_4\right) = \frac{\left|DC\left(DC\left(A_4\right)\right)\right|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \dfrac{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix}}{a_{22}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix}}{a_{23}} \\ \dfrac{\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix}}{a_{32}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix}}{a_{33}} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} = \frac{|FDC\left(DC\left(A_4\right)\right)|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}.$$

Lemma 9. *[2] Consider a square matrix*

$$A_5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}_{5\times 5},$$

of order 5, where $\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$ *is a doubly nonsingular matrix with all nonzero elements. Then*

$$|A_5| = \left(\frac{DC\left(DC\left(A_5\right)\right)}{\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}} \right)^* = \frac{|DC\left(DC\left(DC\left(A_5\right)\right)\right)_1|}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44}. \end{vmatrix}}$$

Lemma 10. *[5] Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $|A| = -|B|$.*

Lemma 11. *[5] Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α, or by multiplying a single column by the scalar α. Then $|A| = \alpha|B|$.*

3. Main Results

Theorem 12. *Given a square matrix* $A_5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}_{5\times 5}$ *of order 5, such that* $\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} > 0$, $\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \neq 0$ *and* $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$, *then*

$$det\,(A_5) = \frac{\begin{vmatrix} \left|[FDC\,(DC\,(A_4))_1]^*\right| & \left|[FDC\,(DC\,(A_4))_2]^*\right| \\ \left|[FDC\,(DC\,(A_4))_3]^*\right| & \left|[FDC\,(DC\,(A_4))_4]^*\right| \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^2}.$$

Proof. By Lemma 7, we have:

$$|A_5| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} = \frac{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} \\ a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}},$$

by Lemma 10, we have

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} \\ a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{15} & a_{12} & a_{13} & a_{14} \\ a_{25} & a_{22} & a_{23} & a_{24} \\ a_{35} & a_{32} & a_{33} & a_{34} \\ a_{45} & a_{42} & a_{43} & a_{44} \end{vmatrix} \\ -\begin{vmatrix} a_{51} & a_{52} & a_{53} & a_{54} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{55} & a_{52} & a_{53} & a_{54} \\ a_{25} & a_{22} & a_{23} & a_{24} \\ a_{35} & a_{32} & a_{33} & a_{34} \\ a_{45} & a_{42} & a_{43} & a_{44} \end{vmatrix} \end{vmatrix},$$

by Lemma 8, we have:

$$|A_5| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}} \cdot \begin{vmatrix} \frac{|DC(DC(A_4))_1|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} & \frac{|DC(DC(A_4))_2|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \\ \frac{|DC(DC(A_4))_3|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} & \frac{|DC(DC(A_4))_4|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \end{vmatrix},$$

using Definition 6 and Lemma 11, we obtain

$$\begin{aligned} |A_5| &= \frac{1}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}} \cdot \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^2} \cdot \begin{vmatrix} \left|[FDC\,(DC\,(A_4))_1]^*\right| & \left|[FDC\,(DC\,(A_4))_2]^*\right| \\ \left|[FDC\,(DC\,(A_4))_3]^*\right| & \left|[FDC\,(DC\,(A_4))_4]^*\right| \end{vmatrix} \\ &= \frac{\begin{vmatrix} \left|[FDC\,(DC\,(A_4))_1]^*\right| & \left|[FDC\,(DC\,(A_4))_2]^*\right| \\ \left|[FDC\,(DC\,(A_4))_3]^*\right| & \left|[FDC\,(DC\,(A_4))_4]^*\right| \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^2}. \end{aligned}$$

This complete the prove. □

Example 1. Let $A = \begin{bmatrix} 2 & 1 & 4 & 1 & 3 \\ 1 & 4 & 3 & 2 & 1 \\ 2 & 5 & 2 & 3 & 2 \\ 1 & 1 & 4 & 1 & 2 \\ 2 & 3 & 1 & 4 & 1 \end{bmatrix}$ be a square matric of order 5, then by using Theorem 12, we have

$$\begin{aligned} |A_5| &= \begin{vmatrix} 2 & 1 & 4 & 1 & 3 \\ 1 & 4 & 3 & 2 & 1 \\ 2 & 5 & 2 & 3 & 2 \\ 1 & 1 & 4 & 1 & 2 \\ 2 & 3 & 1 & 4 & 1 \end{vmatrix} \\ &= \frac{\begin{vmatrix} \left|\begin{bmatrix} 2 & 1 & 4 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 5 & 2 & 3 \\ 1 & 1 & 4 & 1 \end{bmatrix}^*\right| & \left|\begin{bmatrix} 3 & 1 & 4 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 5 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{bmatrix}^*\right| \\ \left|\begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 2 & 5 & 2 & 3 \\ 1 & 1 & 4 & 1 \end{bmatrix}^*\right| & \left|\begin{bmatrix} 1 & 3 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 2 & 5 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{bmatrix}^*\right| \end{vmatrix}}{\begin{vmatrix} 4 & 3 & 2 \\ 5 & 2 & 3 \\ 1 & 4 & 1 \end{vmatrix} \cdot \begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}^2} \\ &= \frac{\begin{vmatrix} 70 & 70 \\ -175 & -350 \end{vmatrix}}{-10 \cdot (-7)^2} \\ &= \frac{-12\,250}{-490} = 25. \end{aligned}$$

Theorem 13. *Given a square matrix*

$$A_6 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}_{6\times 6}$$

of order 6 such that

$$\begin{bmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} > 0, \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \neq 0 \text{ and } \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} \neq 0,$$

then

$$det\left(A_6\right) = \frac{\begin{vmatrix} \left|\left[FDC\left(DC\left(DC\left(A_5\right)\right)\right)_1\right]^*\right| & \left|\left[FDC\left(DC\left(DC\left(A_5\right)\right)\right)_2\right]^*\right| \\ \left|\left[FDC\left(DC\left(DC\left(A_5\right)\right)\right)_3\right]^*\right| & \left|\left[FDC\left(DC\left(DC\left(A_5\right)\right)\right)_4\right]^*\right| \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}^2}.$$

Proof. By Lemma 7, we have

$$\begin{aligned} |A_6| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix} \\ &= \frac{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{55} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{62} & a_{63} & a_{64} & a_{65} & a_{56} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}} \end{aligned}$$

by Lemma 10, we know that

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{55} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{62} & a_{63} & a_{64} & a_{65} & a_{56} \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} & \begin{vmatrix} a_{16} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{26} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{36} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{46} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{56} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \\ \\ \begin{vmatrix} a_{61} & a_{62} & a_{63} & a_{64} & a_{55} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} & \begin{vmatrix} a_{66} & a_{62} & a_{63} & a_{64} & a_{55} \\ a_{26} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{36} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{46} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{56} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \end{vmatrix},$$

using Lemma 9, we have

$$|A_6| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}} \cdot \begin{vmatrix} \dfrac{|DC(DC(DC(A_5)))_1|}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}} & \dfrac{|DC(DC(DC(A_5)))_2|}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}} \\ \dfrac{|DC(DC(DC(A_5)))_3|}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}} & \dfrac{|DC(DC(DC(A_5)))|}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}} \end{vmatrix}$$

using Definition 6 and Lemma 11, we obtain

$$\begin{aligned} |A_6| &= \frac{1}{\begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}} \cdot \frac{1}{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}^2} \\ &\quad \cdot \begin{vmatrix} |[FDC(DC(DC(A_5)))_1]^*| & |[FDC(DC(DC(A_5)))_2]^*| \\ |[FDC(DC(DC(A_5)))_3]^*| & |[FDC(DC(DC(A_5)))_4]^*| \end{vmatrix} \\ &= \frac{\begin{vmatrix} |[FDC(DC(DC(A_5)))_1]^*| & |[FDC(DC(DC(A_5)))_2]^*| \\ |[FDC(DC(DC(A_5)))_3]^*| & |[FDC(DC(DC(A_5)))_4]^*| \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}^2}. \end{aligned}$$

This complete the prove. □

Example 2. Let $A \begin{bmatrix} 1 & 4 & 2 & 1 & 3 & 2 \\ 2 & 1 & 1 & 4 & 2 & 3 \\ 2 & 3 & 4 & 1 & 1 & 2 \\ 3 & 1 & 4 & 2 & 1 & 4 \\ 1 & 4 & 2 & 3 & 4 & 3 \\ 4 & 1 & 1 & 2 & 1 & 3 \end{bmatrix}$ be a square matrix of order 6, then by using Theorem 13, we have

$$|A_6| = \begin{vmatrix} 1 & 4 & 2 & 1 & 3 & 2 \\ 2 & 1 & 1 & 4 & 2 & 3 \\ 2 & 3 & 4 & 1 & 1 & 2 \\ 3 & 1 & 4 & 2 & 1 & 4 \\ 1 & 4 & 2 & 3 & 4 & 3 \\ 4 & 1 & 1 & 2 & 1 & 3 \end{vmatrix} \frac{\begin{vmatrix} \left|\begin{bmatrix} 1 & 4 & 2 & 1 & 3 \\ 2 & 1 & 1 & 4 & 2 \\ 2 & 3 & 4 & 1 & 1 \\ 3 & 1 & 4 & 2 & 1 \\ 1 & 4 & 2 & 3 & 4 \end{bmatrix}^*\right| & \left|\begin{bmatrix} 2 & 4 & 2 & 1 & 3 \\ 3 & 1 & 1 & 4 & 2 \\ 2 & 3 & 4 & 1 & 1 \\ 4 & 1 & 4 & 2 & 1 \\ 3 & 4 & 2 & 3 & 4 \end{bmatrix}^*\right| \\ \left|\begin{bmatrix} 4 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 4 & 2 \\ 2 & 3 & 4 & 1 & 1 \\ 3 & 1 & 4 & 2 & 1 \\ 1 & 4 & 2 & 3 & 4 \end{bmatrix}^*\right| & \left|\begin{bmatrix} 3 & 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & 4 & 2 \\ 2 & 3 & 4 & 1 & 1 \\ 4 & 1 & 4 & 2 & 1 \\ 3 & 4 & 2 & 3 & 4 \end{bmatrix}^*\right| \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 4 & 2 \\ 3 & 4 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 4 & 2 & 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 4 \\ 3 & 4 & 1 \\ 1 & 4 & 2 \end{vmatrix}^2}$$

$$= \frac{\begin{vmatrix} 1426 & 899 \\ 4898 & 2635 \end{vmatrix}}{56 \cdot 31^2} = \frac{-645\,792}{53\,816} = -12.$$

Conflicts of Interest: "The author declare no conflict of interest."

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