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Diversity on $\wp_{\text{fin}}(X)$

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In tight-span theory, a diversity is a generalization of metric space where the metric is defined over the set $\wp_{\text{fin}}(X)$ which is composed of finite subsets of X. In this paper we are going to generalize the results of D. Silvestru and C. Gosa to derive some sharp inequalities for the diameter diversity. This sharp inequality can be used to study models with diversity in a collective manner.

Keywords: Metric spaces; diversity; diameter diversity; sharp inequalities.

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1 Introduction

A diversity or Bryant-Tupper space (BT-space) is a generalization of metric space that have wide applications in non-linear analysis [1]. In this paper, we are going to derive some inequalities for diversities using the results of D. Silvestru and C. Gosa [2].

The study of diversity is also important to study the geometry of hypergraphs. It was shown by Bruant and Tupper in (8) that generalizations of the multi-commodity flow and corresponding minimum cut problems can be used to obtain some results on Steiner Tree Packing and Hypergraph Cut problems using some well known examples of diversity.

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In this paper we have derived some shapr inequalities for diversity in general. This sharp inequality can be used to study models with diversity in a collective manner (9-16). Applications of our main results are also presented in last section.

The paper is organized as follows. In 1st section, we will provide some preliminary definitions that we would be using throughout the paper. In 2nd section, we will derive the main results of this paper and finally in 3rd section, we will apply the main results to diameter diversity.

Let X be a non empty set. Then the ordered pair $\langle X, \varrho \rangle$ is known as a metric space if $\forall x, y, z \in X$, the function $\varrho : X \times X \to \mathbb{R}$ satisfies

- M1 $\varrho(x, y) \geq 0$ and $\varrho(x, y) = 0$ if only if $x = y$.
- M2 $\varrho(x, y) = \varrho(y, x)$.
- M3 $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$.

Bryant and Tupper defined a slightly different mapping, instead of defining ϱ on $X \times X$, they defined it on $\wp_{\text{fin}}(X)$ which is the set of finite subsets of X. Now again, if X is a non empty set, then the ordered pair $\langle X, \delta \rangle$ is known as a diversity if $\forall A, B, C \in \wp_{\text{fin}}(X)$ the function $\delta : \wp_{\text{fin}}(X) \to \mathbb{R}$ satisfies

- D1 $\delta(A) \geq 0$ and $\delta(A) = 0$ if and only if $|A| \leq 1$.
- D2 $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)$ provided that $B \neq \phi$.
- D3 if $A \subseteq B$, then $\delta(A) \leq \delta(B)$.

To avoid using double summations for simplicity of notation purpose, the sum over two indices will be represented under one sum.

2 Main Results

In this section, we are going to derive analogues of theorems derived by D. Silvestru and C. Gosa. For Theorem 1 and Corollary 1, readers can refer to [2] whereas for Theorem 2 readers can refer to [3].

Theorem 1. Let $\langle X, \delta \rangle$ be a diversity and $A, A_i \in \wp_{\text{fin}}(X)$, $p_i \geq 0$ for $i \in \{1, 2, ..., n\}$ such that $\sum_{i=1}^{n} p_i = 1$. Then

$$\sum_{1 \leq i < j \leq n} p_i p_j \delta(A_i \cup A_j) \leq \inf_{A \in \wp_{\text{fin}}(X)} \left[\sum_{1 \leq i \leq n} p_i \delta(A_i \cup A) \right]. \tag{2.1}$$

The above inequality is sharp in the sense that any multiplicative constant $c = 1$ on the right hand side cannot be replaced by a smaller quantity.

Proof. Let $A, A_i, A_j \in \wp_{\text{fin}}(X)$ for $i, j \in \{1, 2, ..., n\}$ provided that $A \neq \phi$. Then, using (D2) we get

$$\delta(A_i \cup A_j) \leq \delta(A_i \cup A) + \delta(A \cup A_j) \tag{2.2}$$

Let $p_i \geq 0$ for $i \in \{1, 2, ..., n\}$ such that $\sum_{i=1}^{n} p_i = 1$. Now multiply $p_i p_j$ on both sides of Eqn.(2.2) and sum on i and j from 1 to n to get

$$\sum_{1 \leq i, j \leq n} p_i p_j \delta(A_i \cup A_j) \leq \sum_{1 \leq i, j \leq n} p_i p_j \left[\delta(A_i \cup A) + \delta(A \cup A_j) \right]. \tag{2.3}$$

We can see that the function δ is symmetric i.e. $\delta(A \cup B) = \delta(B \cup A)$. Therefore, using this property we can rewrite left and right hand side of Eqn. (2.3) as

$$\sum_{1 \leq i, j \leq n} p_i p_j \delta(A_i \cup A_j) = 2 \sum_{1 \leq i < j \leq n} p_i p_j \delta(A_i \cup A_j) \tag{2.4}$$

and

$$\sum_{1\le i,j\le n} p_i p_j \left[\delta\left(A_i\cup A\right)+\delta\left(A\cup A_j\right)\right] = 2\sum_{1\le i\le n} p_i \delta\left(A_i\cup A\right) \tag{2.5}$$

respectively. Therefore, substituting Eqn. (2.4) and (2.5) in Eqn. (2.3) gives

$$\sum_{1\le i<j\le n} p_i p_j \delta\left(A_i\cup A_j\right) \le \sum_{1\le i\le n} p_i \delta\left(A_i\cup A\right). \tag{2.6}$$

Now, taking infimum over A, we get the desired result. Suppose that Eqn. (2.1) is valid for some constant $c>0$, i.e.

$$\sum_{1\le i<j\le n} p_i p_j \delta\left(A_i\cup A_j\right) \le c \inf_{A\in\wp_{\text{fin}}(X)} \left[\sum_{1\le i\le n} p_i \delta\left(A_i\cup A\right)\right]. \tag{2.7}$$

Now, let $n=2, p_1=p, p_2=1-p$ where $p\in(0,1)$. Then, we get $p(p-1)\delta(A_1\cup A_2)\le c\left[p\delta(A_1\cup A)+(p-1)\delta(A\cup A_2)\right]$ where $A_1, A_2\in X$. Now, let $|A_1\cup A|\le 1$. Then, to get $p\delta(A_1\cup A_2)\le c\delta(A_1\cup A_2)$. And since $p\in(0,1)$, the constant c should be greater than on equal to 1, i.e. $c\ge 1$. This implies that Eqn. (2.1) is sharp and any multiplicative constant on right hand side of the equation cannot be replaced by a smaller quantity. $\square$

Following is the corollary of Theorem 1 derived by choosing $p_i=\frac{1}{n}\ \forall i\in\{1,2,...,n\}$.

Corollary 1.1. Let $\langle X,\delta\rangle$ be a diversity and $A, A_i\in X$ for $i\in\{1,2,...,n\}$. Then

$$\sum_{1\le i<j\le n} \delta\left(A_i\cup A_j\right) \le n \inf_{A\in\wp_{\text{fin}}(X)} \left[\sum_{1\le i\le n} \delta\left(A_i\cup A\right)\right]. \tag{2.8}$$

Consider the function $f(t)=t^s$ defined on $[0,\infty)$ for $s\ge 1$. Then, using convexity property of $f(t)$, we get

$$(a+b)^s \le 2^{s-1}(a^s+b^s). \tag{2.9}$$

If $0<s<1$, then we have the following analogue [4]:

$$(a+b)^s \le (a^s+b^s). \tag{2.10}$$

Now, we are going to use above two results in deriving the following theorems.

Theorem 2. Let $\langle X,\delta\rangle$ be a diversity and $A, A_i\in\wp_{\text{fin}}(X)$, $p_i\ge 0$ for $i\in\{1,2,...,n\}$ such that $\sum_{i=1}^n p_i=1$. Then for $s\ge 1$, we have

$$2^{s-1}\left(\sum_{1\le i<j\le n} p_i p_j \delta\left(A_i\cup A_j\right)\right)^2 \le S_1 \le \inf_{A\in\wp_{\text{fin}}(X)} \left[\frac{2^s}{8}\sum_{1\le k\le n} \delta^s\left(A_k\cup A\right)\right] \tag{2.11}$$

where

$$S_1 = \sum_{1\le i<j\le n} p_i p_j \delta^s\left(A_i\cup A_j\right). \tag{2.12}$$

Proof. Let $\langle X,\delta\rangle$ be a diversity and $s\ge 1$. Furthermore, let $A, A_i, A_j\in\wp_{\text{fin}}(X)$ for $i,j\in\{1,2,...,n\}$ provided that $A\ne\phi$. Now, apply Eqn. (2.9) to Eqn. (2.2) to get

$$\delta^s\left(A_i\cup A_j\right) \le 2^{s-1}\left[\delta^s\left(A_i\cup A\right)+\delta^s\left(A\cup A_j\right)\right] \tag{2.13}$$

Let $p_i\ge 0$ for $i\in\{1,2,...,n\}$ such that $\sum_{i=1}^n p_i=1$. Now multiply $p_i p_j$ on both sides of Eqn.(2.13) and sum on i and j over $1\le i<j\le n$ to get

$$\sum_{1\le i<j\le n} p_i p_j \delta^s\left(A_i\cup A_j\right) \le 2^{s-1}\sum_{1\le i<j\le n} p_i p_j\left[\delta^s\left(A_i\cup A\right)+\delta^s\left(A\cup A_j\right)\right]. \tag{2.14}$$

For evaluating above sum, we can rely on the result that if a_{ij} is a symbol such that $a_{ij} = a_{ji}$ where $1 \leq i < j \leq n$, then

$$\sum_{1\leq i<j\leq n} a_{ij} = \frac{1}{2}\left(\sum_{1\leq i,j\leq n} a_{ij} - \sum_{1\leq k\leq n} a_{kk}\right). \tag{2.15}$$

Denote the left hand sum in Eqn. (2.14) by S_1. Then,

$$\begin{aligned} S_1 &\leq \frac{2^{s-1}}{2}\left(\sum_{1\leq i,j\leq n} p_i p_j \left[\delta^s (A_i \cup A) + \delta^s (A \cup A_j)\right] - 2\sum_{1\leq k\leq n} p_k^2 \delta^s (A_k \cup A)\right) \\ &= 2^{s-1}\sum_{1\leq k\leq n} p_k \delta^s (A_k \cup A) - \sum_{1\leq k\leq n} p_k^2 \delta^s (A_k \cup A) \\ &= 2^{s-1}\sum_{1\leq k\leq n} p_k (1-p_k) \delta^s (A_k \cup A) \\ &\leq \frac{2^{s-1}}{4}\sum_{1\leq k\leq n} \delta^s (A_k \cup A). \end{aligned} \tag{2.16}$$

In the last inequality above, we have used the property

$$p_k(1-p_k) \leq \frac{1}{4}(p_k + 1 - p_k)^2 = \frac{1}{4}.$$

Substituting above result in Eqn. (2.14) and taking infimum over A gives

$$\sum_{1\leq i<j\leq n} p_i p_j \delta^s (A_i \cup A_j) \leq \inf_{A\in\wp_{\text{fin}}(X)} \left[\frac{2^s}{8}\sum_{1\leq k\leq n} \delta^s (A_k \cup A)\right]. \tag{2.17}$$

If we use discrete Jensen's inequality on the function $f(t)$ then we get

$$\frac{\sum\limits_{1\leq i,j\leq n} p_i p_j \delta^s (A_i \cup A_j)}{\sum\limits_{1\leq i,j\leq n} p_i p_j} \geq \left(\frac{\sum\limits_{1\leq i,j\leq n} p_i p_j \delta (A_i \cup A_j)}{\sum\limits_{1\leq i,j\leq n} p_i p_j}\right)^s. \tag{2.18}$$

The denominator on both the sides can be found equal to 1 using the definition of p_i. The Numerator on left and right hand side of Eqn. (2.18) can be found equal to

$$\sum_{1\leq i,j\leq n} p_i p_j \delta^s (A_i \cup A_j) = 2\sum_{1\leq i<j\leq n} p_i p_j \delta^s (A_i \cup A_j) \tag{2.19}$$

and

$$\sum_{1\leq i,j\leq n} p_i p_j \delta (A_i \cup A_j) = 2\sum_{1\leq i<j\leq n} p_i p_j \delta (A_i \cup A_j) \tag{2.20}$$

respectively. Substituting above two values in Eqn. (2.18) and simplifying further, we get the following lower bound

$$2^{s-1}\left(\sum_{1\leq i<j\leq n} p_i p_j \delta (A_i \cup A_j)\right)^2 \leq \sum_{1\leq i<j\leq n} p_i p_j \delta^s (A_i \cup A_j). \tag{2.21}$$

□

For the case of $0 < s < 1$, the upper bound of S_1 will be off by the factor of 2^{s-1}. Now, we will apply Theorem 1 and Theorem 2 for deriving results on diameter diversity.

3 Diameter Diversity

Let $\langle X, d\rangle$ be a metric space. For all $A \in \wp_{\text{fin}}(X)$ let

$$\delta(A) = \max_{x,y\in A} d(x,y) = \operatorname{diam} A. \tag{3.1}$$

Then $\langle X, \delta\rangle$ is known as a diameter diversity. Now,

$$\begin{aligned}\sum_{1\le i<j\le n} p_i p_j \delta\left(A_i \cup A_j\right) &= \sum_{1\le i<j\le n} p_i p_j \max_{x,y\in A_i\cup A_j} d(x,y)\\ &= \sum_{1\le i<j\le n} p_i p_j \operatorname{diam}\left(A_i \cup A_j\right).\end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned}\inf_{A\in\wp_{\text{fin}}(X)}\left[\sum_{1\le i\le n} p_i \delta\left(A_i \cup A\right)\right] &= \inf_{A\in\wp_{\text{fin}}(X)}\left[\sum_{1\le i\le n} p_i \max_{x,y\in A_i\cup A} d(x,y)\right]\\ &= \inf_{A\in\wp_{\text{fin}}(X)}\left[\sum_{1\le i\le n} p_i \operatorname{diam}\left(A_i \cup A\right)\right]\end{aligned} \tag{3.3}$$

Now, substituting Eqn. (3.2) and (3.3) in Eqn. (2.1) gives

$$\sum_{1\le i<j\le n} p_i p_j \operatorname{diam}\left(A_i \cup A_j\right) \le \inf_{A\in\wp_{\text{fin}}(X)}\left[\sum_{1\le i\le n} p_i \operatorname{diam}\left(A_i \cup A\right)\right]. \tag{3.5}$$

Similarly, from Theorem 2, we get

$$2^{s-1}\left(\sum_{1\le i<j\le n} p_i p_j \operatorname{diam}\left(A_i \cup A_j\right)\right)^2 \le S_1 \le \inf_{A\in\wp_{\text{fin}}(X)}\left[\frac{2^s}{8}\sum_{1\le k\le n} \operatorname{diam}^s\left(A_k \cup A\right)\right] \tag{3.6}$$

where

$$S_1 = \sum_{1\le i<j\le n} p_i p_j \operatorname{diam}^s\left(A_i \cup A_j\right). \tag{3.7}$$

The results of Silvestru and Gosa from [2] were generalized later and other analogues were found [5]-[9]. This generalized analogues can be also applied to derive further inequalities for diversities on $\wp_{\text{fin}}(X)$.

Some other examples of diversity include the L_1 diversity, Phylogenetic diversity, Steiner diversity, Truncated diversity and Clique diversity.

Let $A \subseteq \mathbb{R}^n$, now if we define

$$\delta(A) = \sum_i \max_{a,b}\left\{|a_i - b_i|, a, b \in A\right\}$$

then the ordered pair $(\mathbb{R}^n, \delta)$ is known as L_1 diversity. If T is a phylogenetic tree with taxon set X. For each finite $A \subseteq X$ define $\delta(A)$ as the length of the smallest subtree of T connecting taxa in A. Then (X, δ) is a (phylogenetic) diversity. Similarly, Steiner diversity corresponds to the Steiner tree.

Diversity has some important and wide applications in theoretical computer science. Bryantand Tupper have shown in [17] that the theory of diversity can be generalized considerably to encompass Steiner tree packing problems in both graphs and hypergraphs.

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Competing Interests

Author has declared that no competing interests exist.

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