Journal of Scientific Research & Reports
3(14): 1927-1972, 2014; Article no. JSRR.2014.14.009

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2-Tuple Linguistic Power Aggregation Operators and Their Application to Multiple Attribute Group Decision Making

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Author's contribution

This whole work was carried out by author ZZ.

Original Research Article

Received 26th April 2014
Accepted 19th May 2014
Published 18th June 2014

ABSTRACT

Aims: The aim of this paper is to develop several 2-tuple linguistic power aggregation operators for aggregating 2-tuple linguistic information.
Study Design: We first introduce several power aggregation operators and then extend these operators to 2-tuple linguistic environments.
Place and Duration of Study: We investigate several useful properties of the developed operators and discuss the relationships between them.
Methodology: Furthermore, the new aggregation operators are utilized to develop two approaches to multiple attribute group decision making with 2-tuple linguistic information.
Results: We develop several 2-tuple linguistic power aggregation operators to aggregate input arguments taking the form of 2-tuples.
Conclusion: Finally, two practical examples are provided to illustrate the effectiveness and practicality of the proposed approaches.

Keywords: Multiple attribute group decision making; 2-tuple linguistic information; Power aggregation operators; 2-Tuple linguistic power aggregation operators.

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1. INTRODUCTION

Multiple attribute group decision making (MAGDM) consists of finding the most desirable alternative(s) from a given alternative set according to the preferences provided by a group of experts [1]. For some MAGDM problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thinking [2-4]; thus, the decision information cannot be precisely assessed in a quantitative form. However, it may be appropriate and sufficient to assess the information in a qualitative form rather than a quantitative form. For example, when evaluating a house's cost, linguistic terms such as ''high'', ''medium'', and ''low'' are usually used, and when evaluating a house's design, linguistic terms like ''good'', ''medium'', and ''bad'' can be frequently used. Up to now, some methods have been developed for dealing with linguistic information [5-17]. In the fuzzy linguistic approach, the results usually do not exactly match any of the initial linguistic terms, then an approximation process must be developed to express the result in the initial expression domain. This produces the consequent loss of information and hence the lack of precision. To overcome this limitation, Herrera and Martinez [18] introduced a 2-tuple fuzzy linguistic representation model that represents the linguistic information by means of 2-tuples, which are composed by a linguistic term and a number [18,19]. The main advantage of this representation is to allow a continuous representation of the linguistic information on its domain, therefore, it can represent any counting of information obtained in a aggregation process without any loss of information [18,19]. In a MAGDM problem with 2-tuple linguistic information, 2-tuple linguistic aggregation operators are most widely used tool for aggregating the individual 2-tuple linguistic information into the overall 2-tuple linguistic information. In the past few decades, a variety of 2-tuple linguistic aggregation operators have been developed, including 2-tuple arithmetic mean operator [18,20], 2-tuple weighted averaging operator [18], 2-tuple OWA operator [18], 2-tuple weighted geometric averaging (TWGA) operator [21], 2-tuple ordered weighted geometric averaging (TOWGA) operator [21], 2-tuple hybrid geometric averaging (THGA) operator [21], 2-tuple arithmetic average (TAA) operator [18], 2-tuple weighted average (TWA) operator [18], 2-tuple ordered weighted average (TOWA) operator [18], extended 2-tuple weighted average (ET-WA) operator [18], 2-tuple ordered weighted geometric (TOWG) operator [22,23], extended 2-tuple weighted geometric (ET-WG) operator [23], extended 2-tuple ordered weighted geometric (ET-OWG) operator [23], generalized 2-tuple weighted average (G-2TWA) operator [24], generalized 2-tuple ordered weighted average (G-2TOWA) operator [24], and induced generalized 2-tuple ordered weighted average (IG-2TOWA) operator [24]. However, the aforementioned 2-tuple linguistic aggregation operators do not consider information about the relationship between the values being fused.

Yager [25] introduced the power average (PA) and power ordered weighted average (POWA) operators, which are two nonlinear weighted average aggregation tools whose weight vectors depend on the input arguments and which allow the values being aggregated to support and reinforce one another [26]. Motivated by the success of the PA and POWA operators, based on the PA operator and the geometric mean, Xu and Yager [26] developed the power geometric (PG) operator. Based on the POWA operator and the geometric mean, Xu and Yager [26] developed the power ordered weighted geometric (POWG) operator. Zhou et al. [27] investigated a generalized power average (GPA) operator and its weighted form by combining the PA operator with the generalized mean [28], and developed a generalized power ordered weighted average (GPOWA) operator by combining the POWA operator with the generalized mean. Furthermore, many scholars extended the power aggregation operators to other domains and developed numerous extensions of the aforementioned power aggregation operators, such as the uncertain power geometric (UPG) operator and its weighted form [26], uncertain power ordered weighted geometric (UPOWG) operator [26], uncertain generalized power average (UGPA) operator [27], uncertain weighted generalized power average (UWGPA) operator [27], uncertain generalized power ordered weighted average (UGPOWA) operator [27], uncertain power ordered weighted average (UPOWA) operator [29], intuitionistic fuzzy power weighted average (IFPWA) operator [30], intuitionistic fuzzy power average (IFPA) operator [30], intuitionistic fuzzy power weighted geometric (IFPWG) operator [30], intuitionistic fuzzy power geometric (IFPG) operator [30], intuitionistic fuzzy power ordered weighted average (IFPOWA) operator [30], intuitionistic fuzzy power ordered weighted geometric (IFPOWG) operator [30], generalized intuitionistic fuzzy power averaging(GIFPA) operator [27], generalized intuitionistic fuzzy power ordered weighted averaging (GIFPOWA) operator [27], interval-valued intuitionistic fuzzy power weighted average (IVIFPWA) operator [30], interval-valued intuitionistic fuzzy power average (IVIFPA) operator [30], interval-valued intuitionistic fuzzy power weighted geometric (IVIFPWG) operator [30], interval-valued intuitionistic fuzzy power geometric (IVIFPG) operator [30], interval-valued intuitionistic fuzzy power ordered weighted average (IVIFPOWA) operator [30], interval-valued intuitionistic fuzzy power ordered weighted geometric (IVIFPOWG) operator [30], linguistic power average (LPA) operator [31], linguistic weighted power average (LWPA) operator [31], linguistic power ordered weighted averaging (LPOWA) operator [31], linguistic generalized power average (LGPA) operator [32], weighted linguistic generalized power average (WLGPA) operator [32], linguistic generalized power ordered weighted average (LGPOWA) operator [32], uncertain linguistic power averaging (ULPA) operator [31], uncertain linguistic weighted power average (ULWPA) operator [31], and uncertain linguistic power ordered weighted averaging (ULPOWA) operator [31]. However, these power aggregation operators cannot accommodate situations where the input arguments take the form of 2-tuples.

Based on the above analysis, we can conclude that the existing 2-tuple linguistic aggregation operators do not consider information about the relationship between the values being fused and the existing power aggregation operators cannot accommodate situations where the input arguments take the form of 2-tuples. To address this issue, it is therefore necessary to develop some new aggregation operators that not only accommodate 2-tuple linguistic information but also consider the information about the relationship between the values being fused. We are only aware of one paper on the combination of power aggregation operators and 2-tuple linguistic aggregation operators. In [33], Xu and Wang extended the power average (PA) and power ordered weighted average (POWA) operators to 2-tuple linguistic environments and developed several 2-tuple linguistic power average aggregation operators for aggregating 2-tuple linguistic information, including 2-tuple linguistic power average (2TLPA) operator, 2-tuple linguistic power weighted average (2TLPWA) operator, and 2-tuple linguistic power ordered weighted average (2TLPOWA) operator. The primary characteristic of these operators is that they not only accommodate input arguments in the form of 2-tuples but also incorporate information regarding the relationship between the values being combined. This paper contains three parts: (1) First, we investigate a generalized 2-tuple linguistic power average (G2TLPA) operators and its weighted form (a generalized 2-tuple linguistic power weighted average (G2TLPWA) operator), which are on the basis of the 2TLPA operator and the generalized mean, and develop a generalized 2-tuple linguistic power ordered weighted average (G2TLPOWA) operator, which is on the basis of the 2TLPOWA operator and the generalized mean, and study some of their properties. These newly developed generalized 2-tuple linguistic power average operators add to the Xu and Wang's 2-tuple linguistic power average operators an additional parameter controlling the power to which the argument values are raised. When we use different choices for the parameter λ, we obtain some special cases. The Xu and Wang's 2-tuple linguistic power average operators are special cases of these newly developed generalized 2-tuple linguistic power average operators. (2) Furthermore, we develop a 2-tuple linguistic power geometric (2TLPG) operator and its weighted form (a 2-tuple linguistic power weighted geometric (2TLPWG) operator), which are on the basis of the 2TLPA operator and the geometric mean, and develop a 2-tuple linguistic power ordered weighted geometric (2TLPOWG) operator, which is on the basis of the 2TLPOWA operator and the geometric mean, and study some of their properties. We also discuss the relationship between the 2TLPG and G2TLPA operators, the relationship between the 2TLPWG and G2TLPWA operators, and the relationship between the 2TLPOWG and G2TLPOWA operators. The Xu and Wang's 2-tuple linguistic power average operators are based on the arithmetic average tool, which is one of the basic aggregation techniques and which focuses on the group opinion. These newly developed 2-tuple linguistic power geometric operators are based on the geometric mean, which gives more importance to individual opinions. (3) Finally, we utilize the proposed operators to develop two approaches to multiple attribute group decision making with 2-tuple linguistic information and then apply both the developed approaches to two practical examples.

In order to do that, this paper is organized as follows. In Section 2, we briefly review some basic concepts of the 2-tuple fuzzy linguistic approach, power aggregation operators, and 2-tuple linguistic power aggregation operators. In Section 3, we present several new 2-tuple linguistic power aggregation operators, investigate some of their basic properties, and discuss the relationships between the various operators. Section 4 develops two approaches to multiple attribute group decision making with 2-tuple linguistic information based on the proposed operators. Two illustrative examples are provided in Section 5. Finally, we summarize the paper in Section 6.

2. PRELIMINARIES

In this section, we will introduce the basic notions of the 2-tuple fuzzy linguistic approach and power aggregation operators. Let R be a set of all real numbers.

2.1 The 2-Tuple Fuzzy Linguistic Representation Model

Let $S=\{s_i | i=0,1,2,\cdots,g\}$ be a finite and totally ordered discrete linguistic term set with odd cardinality, where s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics [18,20,34,35].

(1) The set is ordered: $s_i \geq s_j$ if $i \geq j$;
(2) There is the negation operator: $neg(s_i)=s_j$ such that $j=g-i$;
(3) Max operator: $\max(s_i,s_j)=s_i$ if $s_i \geq s_j$;
(4) Min operator: $\min(s_i,s_j)=s_i$ if $s_i \leq s_j$.

For example, a set of seven terms S could be given as follows [35-39]:

$$S = \{s_0 = nothing,\ s_1 = very\,low,\ s_2 = low,\ s_3 = medium,\ s_4 = high,\ s_5 = very\,high,\ s_6 = perfect\}$$

Based on the concept of symbolic translation, Herrera and Martinez [18,34] introduced a 2-tuple fuzzy linguistic representation model for dealing with linguistic information. This model represents the linguistic assessment information by means of a 2-tuple (s_i, α), where $s_i \in S$ represents a linguistic label from the predefined linguistic term set S and $\alpha \in [-0.5, 0.5)$ is the value of symbolic translation.

Definition 2.1 [18,34]. Let β be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set S, i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, being $g+1$ the cardinality of S. Let $i = \mathrm{round}(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$ then α is called a symbolic translation, where $\mathrm{round}(\cdot)$ is the usual round operation.

Definition 2.2 [18,34]. Let $S = \{s_i | i = 0, 1, 2, \cdots, g\}$ be a linguistic term set and $\beta \in [0, g]$ a value representing the result of a symbolic aggregation operation. Then, the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0, g] \to S \times [-0.5, 0.5) \tag{1}$$

$$\Delta(\beta) = (s_i, \alpha),$$

$$\text{with } \begin{cases} s_i, & i = \mathrm{round}(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases} \tag{2}$$

where s_i has the closest index label to β and α is the value of the symbolic translation.

Theorem 2.1 [18,34]. Let $S = \{s_i | i = 0, 1, 2, \cdots, g\}$ be a linguistic term set and (s_i, α) be a 2-tuple. There is always a Δ^{-1} function such that from a 2-tuple it returns its equivalent numerical value $\beta \in [0, g] \subset R$, where

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, g] \tag{3}$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta \tag{4}$$

It is obvious that the conversion of a linguistic term into a linguistic 2-tuple consist of adding a value zero as symbolic translation

$$s_i \in S \Rightarrow (s_i, 0).$$

Definition 2.3 [18,34]. The comparison of linguistic information represented by 2-tuples is carried out according to an ordinary lexicographic order. Let (s_k, α_k) and (s_l, α_l) be two 2-tuples, with each one representing a counting of information as follows.

(1) If $k < l$ then (s_k, α_k) is smaller than (s_l, α_l).
(2) If $k = l$ then

- if $\alpha_k = \alpha_l$ then (s_k, α_k), (s_l, α_l) represents the same information;
- if $\alpha_k < \alpha_l$ then (s_k, α_k) is smaller than (s_l, α_l);
- if $\alpha_k > \alpha_l$ then (s_k, α_k) is bigger than (s_l, α_l).

Theorem 2.2. Let (s_k, α_k) and (s_l, α_l) be two 2-tuples, $\beta_k = \Delta^{-1}(s_k, \alpha_k)$, and $\beta_l = \Delta^{-1}(s_l, \alpha_l)$. Then, $(s_k, \alpha_k) < (s_l, \alpha_l)$ if and only if $\beta_k < \beta_l$, and $(s_k, \alpha_k) = (s_l, \alpha_l)$ if and only if $\beta_k = \beta_l$.

Proof. (1) We first prove that $(s_k, \alpha_k) < (s_l, \alpha_l)$ if and only if $\beta_k < \beta_l$. Assume that $(s_k, \alpha_k) < (s_l, \alpha_l)$. Then, $k < l$, or $k = l$ and $\alpha_k < \alpha_l$. If $k < l$, then we have $\beta_k = k + \alpha_k < k + 0.5 \le l - 0.5 \le l + \alpha_l = \beta_l$. If $k = l$ and $\alpha_k < \alpha_l$, then we have $\beta_k = k + \alpha_k < l + \alpha_l = \beta_l$.

Assume that $\beta_k < \beta_l$. Then, $k < l$, or $k = l$ and $\alpha_k < \alpha_l$. If $k < l$, then we have $(s_k, \alpha_k) < (s_l, \alpha_l)$. If $k = l$ and $\alpha_k < \alpha_l$, then we have $(s_k, \alpha_k) < (s_l, \alpha_l)$.

(2) We next prove that $(s_k, \alpha_k) = (s_l, \alpha_l)$ if and only if $\beta_k = \beta_l$. If $(s_k, \alpha_k) = (s_l, \alpha_l)$, then $k = l$ and $\alpha_k = \alpha_l$, which implies that $\beta_k = k + \alpha_k = l + \alpha_l = \beta_l$. If $\beta_k = \beta_l$, then $k = l$ and $\alpha_k = \alpha_l$, which implies that $(s_k, \alpha_k) = (s_l, \alpha_l)$.

2.2 Power Aggregation Operators

In this subsection, we first briefly review the power average (PA) operator and the power ordered weighted average (POWA) operator [25].

Definition 2.4 [25]. The power average (PA) operator is the mapping $\mathrm{PA}: R^n \to R$ defined by the following formula:

$$\mathrm{PA}(a_1,a_2,\ldots,a_n)=\frac{\sum_{i=1}^{n}\left(1+T(a_i)\right)a_i}{\sum_{i=1}^{n}\left(1+T(a_i)\right)} \tag{5}$$

Where

$$T(a_i)=\sum_{\substack{j=1\\ j\neq i}}^{n} Supp(a_i,a_j) \tag{6}$$

and $Supp(a_i,a_j)$ is the support for a_i from a_j. The support satisfies the following three properties:

(1) $Supp(a_i,a_j)\in[0,1]$.
(2) $Supp(a_i,a_j)=Supp(a_j,a_i)$.
(3) If $|a_i-a_j|<|a_s-a_t|$, then $Supp(a_i,a_j)\geq Supp(a_s,a_t)$.

Definition 2.5 [25]. The power ordered weighted average (POWA) operator is the mapping $\mathrm{POWA}:R^n\rightarrow R$ defined by the following formula:

$$\mathrm{POWA}(a_1,a_2,\ldots,a_n)=\sum_{i=1}^{n}u_i a_{index(i)}, \tag{7}$$

Where

$$u_i=g\left(\frac{R_i}{TV}\right)-g\left(\frac{R_{i-1}}{TV}\right),\quad R_i=\sum_{j=1}^{i}V_{index(j)},\quad TV=\sum_{i=1}^{n}V_{index(i)},\quad V_{index(i)}=1+T\left(a_{index(i)}\right),$$

$$T\left(a_{index(i)}\right)=\sum_{\substack{j=1\\ j\neq i}}^{n} Supp\left(a_{index(i)},a_{index(j)}\right) \tag{8}$$

In the above equation, $a_{index(i)}$ is the *i*th largest argument of all the arguments $a_j\ (j=1,2,\ldots,n)$, $T\left(a_{index(i)}\right)$ denotes the support of the *i*th largest argument by all the other arguments, $Supp\left(a_{index(i)},a_{index(j)}\right)$ indicates the support of the *j*th largest argument for the *i*th largest argument, and $g:[0,1]\rightarrow[0,1]$ is a basic unit-interval monotonic (BUM) function that has the following properties: (1) $g(0)=0$, (2) $g(1)=1$, and (3) if $x>y$, then $g(x)\geq g(y)$. Motivated by Yager [25] and based on the PA operator and the geometric mean, Xu and Yager [26] defined the power geometric (PG) operator:

Definition 2.6 [26]. The PG operator is the mapping $\mathrm{PG}: R^n \to R$ defined by the following formula:

$$\mathrm{PG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i^{\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}}, \tag{9}$$

Where $a_i\,(i=1,2,\ldots,n)$ is a collection of arguments and $T(a_i)$ satisfies Eq. (6).

Based on the POWA operator and the geometric mean, Xu and Yager [26] defined a power ordered weighted geometric (POWG) operator as follows:

Definition 2.7 [26]. The POWG operator is the mapping $\mathrm{POWG}: R^n \to R$ defined by the following formula:

$$\mathrm{POWG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_{index(i)}^{u_i} \tag{10}$$

in which u_i satisfies Eq. (8) and $a_{index(i)}$ is the *i*th largest argument of $a_j\,(j=1,2,\ldots,n)$.

2.3 The Existing 2-Tuple Linguistic Power Aggregation Operator

Definition 2.8 [33]. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i \in S, \alpha_i \in [-0.5,0.5), i=1,2,\cdots,n)$ be a collection of 2-tuples. If

$$2\mathrm{TLPA}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)) = \Delta\left(\frac{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))\Delta^{-1}(r_i,\alpha_i)}{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\right) \tag{11}$$

then $2\mathrm{TLPA}$ is called a 2-tuple linguistic power average (2TLPA) operator, where

$$T(r_i,\alpha_i) = \sum_{\substack{j=1\\ j\neq i}}^{n} Sup((r_i,\alpha_i),(r_j,\alpha_j)) \tag{12}$$

and $Sup((r_i,\alpha_i),(r_j,\alpha_j))$ is the support for (r_i,α_i) from (r_j,α_j), which satisfies the following three properties:

(1) $Sup((r_i,\alpha_i),(r_j,\alpha_j)) \in [0,1]$;

(2) $Sup((r_i,\alpha_i),(r_j,\alpha_j)) = Sup((r_j,\alpha_j),(r_i,\alpha_i))$;

(3) $Sup\left(\left(r_i,\alpha_i\right),\left(r_j,\alpha_j\right)\right)\geq Sup\left(\left(r_s,\alpha_s\right),\left(r_t,\alpha_t\right)\right)$, if $d\left(\left(r_i,\alpha_i\right),\left(r_j,\alpha_j\right)\right)<d\left(\left(r_s,\alpha_s\right),\left(r_t,\alpha_t\right)\right)$, where d is a distance measure between two linguistic variables.

Definition 2.9 [33]. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples. If

$$\mathrm{2TLPWA}_w\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)=\Delta\left(\frac{\sum_{i=1}^{n}w_i\left(1+T\left(r_i,\alpha_i\right)\right)\Delta^{-1}\left(r_i,\alpha_i\right)}{\sum_{i=1}^{n}w_i\left(1+T\left(r_i,\alpha_i\right)\right)}\right) \tag{13}$$

Then $\mathrm{2TLPWA}$ is called a 2-tuple linguistic power weighted average (2TLPWA) operator, where $T\left(r_i,\alpha_i\right)$ satisfies Eq. (12), $w_i\in[0,1]$ for $i=1,2\cdots,n$, and $\sum_{i=1}^{n}w_i=1$.

Definition 2.10 [33]. For a collection of 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_j\in S,\alpha_j\in[-0.5,0.5),j=1,2,\cdots,n\right)$, a 2-tuple linguistic power ordered weighted average (2TLPOWA) operator is a mapping $H^n\rightarrow H$ such that

$$\mathrm{2TLPOWA}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)=\Delta\left(\sum_{i=1}^{n}\left(u_i\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)\right), \tag{14}$$

Where

$$u_i=g\left(\frac{R_i}{TV}\right)-g\left(\frac{R_{i-1}}{TV}\right),\quad R_i=\sum_{j=1}^{i}V_{index(j)},\quad TV=\sum_{i=1}^{n}V_{index(i)},$$
$$V_{index(i)}=1+T\left(r_{index(i)},\alpha_{index(i)}\right),$$
$$T\left(r_{index(i)},\alpha_{index(i)}\right)=\sum_{\substack{j=1\\ j\neq i}}^{n}Sup\left(\left(r_{index(i)},\alpha_{index(i)}\right),\left(r_{index(j)},\alpha_{index(j)}\right)\right). \tag{15}$$

In Eq. (15), $\left(r_{index(i)},\alpha_{index(i)}\right)$ is the *i*th largest 2-tuple among all of the 2-tuples, $\left(r_j,\alpha_j\right)$ $\left(j=1,2\cdots,n\right)$, $T\left(r_{index(i)},\alpha_{index(i)}\right)$ denotes the support of the *i*th largest 2-tuple by all of the other 2-tuples, $Sup\left(\left(r_{index(i)},\alpha_{index(i)}\right),\left(r_{index(j)},\alpha_{index(j)}\right)\right)$ denotes the support of the *j*th largest 2-tuple for the *i*th largest 2-tuple, and $g:[0,1]\rightarrow[0,1]$ is a basic unit-interval monotonic (BUM) function having the following properties: (1) $g(0)=0$, (2) $g(1)=1$, and (3) $g(x)\geq g(y)$ if $x>y$.

2.4 Tuple Linguistic Power Aggregation Operators

In this section, we first extend the PG operator (Eq. (9)) to 2-tuple environment, i.e., develop a 2-tuple linguistic power geometric (2TLPG) operator and its weight form, which are carried out in Subsection 3.1. Then, we extend the POWG operator (Eq. (10)) to 2-tuple environment, i.e., develop a 2-tuple linguistic power ordered weighted geometric (2TLPOWG) operator, which is conducted in Subsection 3.2. Furthermore, Subsection 3.3 adds an additional parameter λ to the 2TLPA operator (Eq. (11)) and its weighted form (Eq. (13)), i.e., develops a generalized 2-tuple linguistic power average (G2TLPA) operator and its weighted form. Finally, Subsection 3.4 adds an additional parameter λ to the 2TLPOWA operator (Eq. (14)) and develops a generalized 2-tuple linguistic power ordered weighted average (G2TLPOWA) operator.

2.5 Tuple Linguistic Power Geometric (2TLPG) Operators and 2-Tuple Linguistic Power Weighted Geometric (2TLPWG) Operators

Definition 3.1. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i \in S, \alpha_i \in [-0.5,0.5), i=1,2,\cdots,n\right)$ be a collection of 2-tuples. If

$$\text{2TLPG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\left(1+T(r_i,\alpha_i)\right)\big/\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}\right), \quad (16)$$

Then 2TLPG is called a 2-tuple linguistic power geometric (2TLPG) operator, where

$$T(r_i,\alpha_i)=\sum_{\substack{j=1\\ j\neq i}}^{n} Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right) \quad (17)$$

and $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right)$ is the support for (r_i,α_i) from (r_j,α_j), which satisfies the following three properties:

(1) $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right)\in[0,1]$;

(2) $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right)=Sup\left((r_j,\alpha_j),(r_i,\alpha_i)\right)$;

(3) $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right)\geq Sup\left((r_s,\alpha_s),(r_t,\alpha_t)\right)$, if $d\left((r_i,\alpha_i),(r_j,\alpha_j)\right)< d\left((r_s,\alpha_s),(r_t,\alpha_t)\right)$,

where d is a distance measure between two linguistic variables.

Clearly, the support (i.e., Sup) measure is essentially a similarity measure, which can be used to measure the proximity of a preference value provided by one decision maker to another one provided by a different decision maker. The higher the similarity, the smaller the distance between the two linguistic variables and the more they support each other. The 2TLPG operator is a nonlinear weighted aggregation tool, and the weight

$\frac{1+T(r_i,\alpha_i)}{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}$ of argument (r_i,α_i) depends on all of the input arguments $(r_j,\alpha_j)(j=1,2,\cdots,n)$ and allows the argument values to support each other in the aggregation process.

Theorem 3.1. Let $Sup((r_i,\alpha_i),(r_j,\alpha_j))=k$ for all $i\neq j$. Then,

$$\text{2TLPG}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{1/n}\right), \quad (18)$$

indicating that when all of the supports are the same, the 2TLPG operator reduces to the 2-tuple geometric (2TG) operator [22].

Proof. If $Sup((r_i,\alpha_i),(r_j,\alpha_j))=k$ for all $i\neq j$, then

$$T(r_i,\alpha_i)=\sum_{\substack{j=1\\ j\neq i}}^{n}Sup((r_i,\alpha_i),(r_j,\alpha_j))=(n-1)k.$$

We therefore have

$$\begin{aligned}\text{2TPG}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{(1+T(r_i,\alpha_i))/\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\right)\\&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{(1+(n-1)k)/\sum_{i=1}^{n}(1+(n-1)k)}\right)\\&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{1/n}\right),\end{aligned}$$

which is simply a 2-tuple geometric (2TG) operator [22].

Theorem 3.2. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n)$ be a collection of 2-tuples. Then, the following properties hold.

(1) Commutativity: If $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, then

$$\text{2TLPG}((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n))=\text{2TLPG}((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')). \quad (19)$$

(2) Idempotency: If $(r_i,\alpha_i)=(r,\alpha)$ for all i, then

$$\mathrm{2TLPG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)=\left(r,\alpha\right). \tag{20}$$

(3) Boundedness:

$$\min_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}\le \mathrm{2TLPG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)\le \max_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}. \tag{21}$$

Proof. (1) Suppose that $\left\{\left(r_1',\alpha_1'\right),\left(r_2',\alpha_2'\right),\cdots,\left(r_n',\alpha_n'\right)\right\}$ is any permutation of $\left\{\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right\}$. Then, for each $\left(r_i,\alpha_i\right)$, there exists one and only one $\left(r_j',\alpha_j'\right)$ such that $\left(r_i,\alpha_i\right)=\left(r_j',\alpha_j'\right)$ and vice versa. We have $T\left(r_i,\alpha_i\right)=T\left(r_j',\alpha_j'\right)$, and therefore,

$$\begin{aligned}\mathrm{2TLPG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(r_i,\alpha_i\right)\right)^{\left(1+T\left(r_i,\alpha_i\right)\right)/\sum_{i=1}^{n}\left(1+T\left(r_i,\alpha_i\right)\right)}\right)\\&=\Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_j',\alpha_j'\right)\right)^{\left(1+T\left(r_j',\alpha_j'\right)\right)/\sum_{i=1}^{n}\left(1+T\left(r_j',\alpha_j'\right)\right)}\right)\\&=\mathrm{2TLPG}\left(\left(r_1',\alpha_1'\right),\left(r_2',\alpha_2'\right),\cdots,\left(r_n',\alpha_n'\right)\right).\end{aligned}$$

(2) If $\left(r_i,\alpha_i\right)=\left(r,\alpha\right)$ for all i, then

$$\begin{aligned}\mathrm{2TLPG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(r_i,\alpha_i\right)\right)^{\left(1+T\left(r_i,\alpha_i\right)\right)/\sum_{i=1}^{n}\left(1+T\left(r_i,\alpha_i\right)\right)}\right)\\&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(r,\alpha\right)\right)^{\left(1+T\left(r,\alpha\right)\right)/\sum_{i=1}^{n}\left(1+T\left(r,\alpha\right)\right)}\right)\\&=\left(r,\alpha\right).\end{aligned}$$

(3) Because $\min_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}\le \mathrm{2TLPG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)\le \max_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}$, we have

$$\begin{aligned}\min_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(\min_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}\right)\right)^{\left(1+T\left(r_i,\alpha_i\right)\right)/\sum_{i=1}^{n}\left(1+T\left(r_i,\alpha_i\right)\right)}\right)\\&\le\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(r_i,\alpha_i\right)\right)^{\left(1+T\left(r_i,\alpha_i\right)\right)/\sum_{i=1}^{n}\left(1+T\left(r_i,\alpha_i\right)\right)}\right)\\&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(\max_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}\right)\right)^{\left(1+T\left(r_i,\alpha_i\right)\right)/\sum_{i=1}^{n}\left(1+T\left(r_i,\alpha_i\right)\right)}\right)\\&=\max_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}.\end{aligned}$$

The proof of Theorem 3.2 is complete.

Lemma 3.1 [40,41]. Let $x_i > 0$, $\lambda_i > 0$, $i = 1, 2, \cdots, n$, and $\sum_{i=1}^{n} \lambda_i = 1$. Then,

$$\prod_{i=1}^{n} (x_i)^{\lambda_i} \le \sum_{i=1}^{n} \lambda_i x_i$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 3.3. Suppose that $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i \in S, \alpha_i \in [-0.5, 0.5), i = 1, 2, \cdots, n\right)$ is a collection of 2-tuples. Then, we have

$$\text{2TLPG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \le \text{2TLPA}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (22)$$

Proof. Because $\sum_{i=1}^{n} \frac{1+T(r_i,\alpha_i)}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)} = \frac{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)} = 1$, we have by Lemma 3.1:

$$\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\left(1+T(r_i,\alpha_i)\right)/\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)} \le \sum_{i=1}^{n}\left(\frac{\left(1+T(r_i,\alpha_i)\right)\Delta^{-1}(r_i,\alpha_i)}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}\right).$$

Therefore, by Theorem 2.2, we have

$$\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\left(1+T(r_i,\alpha_i)\right)/\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}\right) \le \Delta\left(\sum_{i=1}^{n}\left(\frac{\left(1+T(r_i,\alpha_i)\right)\Delta^{-1}(r_i,\alpha_i)}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}\right)\right),$$

Which implies that

$$\text{2TLPG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \le \text{2TLPA}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right).$$

Theorem 3.3 shows that the values obtained with the 2TLPG operator are not larger than those obtained with the 2TLPA operator.

In the 2TLPG operator, all of the arguments that are being aggregated are of equal importance. If we allow the arguments to have different weights, then the 2-tuple linguistic power weighted geometric (2TLPWG) operator can be defined as follows:

Definition 3.2. For a collection of 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i \in S, \alpha_i \in [-0.5, 0.5), i = 1, 2, \cdots, n\right)$, a 2-tuple linguistic power weighted geometric (2TLPWG) operator is a mapping $H^n \to H$ such that

$$\mathrm{2TLPWG}_w\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\left(w_i\left(1+T'(r_i,\alpha_i)\right)\right)/\sum_{i=1}^{n}\left(w_i\left(1+T'(r_i,\alpha_i)\right)\right)}\right) \quad (23)$$

Where

$$T'(r_i,\alpha_i)=\sum_{\substack{j=1\\ j\neq i}}^{n} w_j Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right) \quad (24)$$

with the conditions that $w=(w_1,w_2,\cdots,w_n)^T$, $w_i\in[0,1]$ for $i=1,2\cdots,n$, and $\sum_{i=1}^{n} w_i=1$.

In particular, if $w=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$, then the 2TLPWG operator reduces to the 2TLPG operator, that is, when all of the arguments are of equal importance, the 2TLPWG operator should be 2TLPG operator.

Theorem 3.4. Let $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right)=k$ for all $i\neq j$. Then,

$$\mathrm{2TLPWG}_w\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{w_i}\right), \quad (25)$$

indicating that when all of the supports are the same, the 2TLPWG operator reduces to the 2-tuple weighted geometric (2TWG) operator [22].

Proof. If $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right)=k$ for all $i\neq j$, then

$$T(r_i,\alpha_i)=\sum_{\substack{j=1\\ j\neq i}}^{n} Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right)=(n-1)k.$$

We therefore have

$$\begin{aligned}\mathrm{2TLPWG}_w\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\left(w_i\left(1+T(r_i,\alpha_i)\right)\right)/\sum_{i=1}^{n}\left(w_i\left(1+T(r_i,\alpha_i)\right)\right)}\right)\\&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\left(w_i\left(1+(n-1)k\right)\right)/\sum_{i=1}^{n}\left(w_i\left(1+(n-1)k\right)\right)}\right)\\&=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{w_i}\right),\end{aligned}$$

which is simply a 2-tuple weighted geometric (2TWG) operator [22].

Similar to the 2TLPG operator, the 2TLPWG operator has the properties such as idempotency and boundedness, but commutativity property does not hold. In fact, if $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, then

$$T'(r_i',\alpha_i')=\sum_{\substack{j=1\\ j\neq i}}^{n} w_j Sup\left((r_i',\alpha_i'),(r_j',\alpha_j')\right)$$

and thus

$$\text{2TLPWG}_w\left((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\right)=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i',\alpha_i')\right)^{\left(w_i\left(1+T'(r_i',\alpha_i')\right)\right)\Big/\sum_{i=1}^{n}\left(w_i\left(1+T'(r_i',\alpha_i')\right)\right)}\right).$$

Since $\{T'(r_1',\alpha_1'),T'(r_2',\alpha_2'),\cdots,T'(r_n',\alpha_n')\}$ may not be the permutation of $\{T'(r_1,\alpha_1),T'(r_2,\alpha_2),\cdots,T'(r_n,\alpha_n)\}$,then $\text{2TLPWG}_w\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\text{2TLPWG}_w\left((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\right)$ generally does not hold.

Similar to the 2TLPG operator, the 2TLPWG operator has the following property:

Theorem 3.5. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples and assume that $\lambda>0$. Then,

$$\text{2TLPWG}_w\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\leq\text{2TLPWA}_w\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (26)$$

Theorem 3.5 shows that the values obtained with the 2TLPWG operator are not larger than those obtained with the 2TLPWA operator for any $\lambda>0$.

2.6 Tuple Linguistic Power Ordered Weighted Geometric (2TLPOWG) Operators

Based on the POWG and 2TLPG operators, we next define a 2-tuple linguistic power ordered weighted geometric (2TLPOWG) operator as follows.

Definition 3.3. For a collection of 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$, a 2-tuple linguistic power ordered weighted geometric (2TLPOWG) operator is a mapping $H^n\to H$ such that

$$\text{2TLPOWG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)^{u_i}\right), \quad (27)$$

Where

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right),\ R_i = \sum_{j=1}^{i} V_{index(j)},\ TV = \sum_{i=1}^{n} V_{index(i)},\ V_{index(i)} = 1 + T\left(r_{index(i)}, \alpha_{index(i)}\right),$$

$$T\left(r_{index(i)}, \alpha_{index(i)}\right) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Sup\left(\left(r_{index(i)}, \alpha_{index(i)}\right), \left(r_{index(j)}, \alpha_{index(j)}\right)\right). \tag{28}$$

In Eq. (28), $\left(r_{index(i)}, \alpha_{index(i)}\right)$ is the *i*th largest 2-tuple among all of the 2-tuples, $\left(r_j, \alpha_j\right)$ $(j = 1, 2 \cdots, n)$, $T\left(r_{index(i)}, \alpha_{index(i)}\right)$ denotes the support of the *i*th largest 2-tuple by all of the other 2-tuples, $Sup\left(\left(r_{index(i)}, \alpha_{index(i)}\right), \left(r_{index(j)}, \alpha_{index(j)}\right)\right)$ denotes the support of the *j*th largest 2-tuple for the *i*th largest 2-tuple, and $g:[0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function having the following properties: (1) $g(0) = 0$, (2) $g(1) = 1$, and (3) $g(x) \geq g(y)$ if $x > y$.

In particular, if $g(x) = x$, then the 2TLPOWG operator reduces to the 2TLPG operator. By Eq. (27), we have

$$\begin{aligned}
\text{2TLPOWG}\left((r_1, \alpha_1), (r_2, \alpha_2), \cdots, (r_n, \alpha_n)\right) &= \Delta\left(\prod_{i=1}^{n} \left(\Delta^{-1}\left(r_{index(i)}, \alpha_{index(i)}\right)\right)^{u_i}\right) \\
&= \Delta\left(\prod_{i=1}^{n} \left(\Delta^{-1}\left(r_{index(i)}, \alpha_{index(i)}\right)\right)^{\left(g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right)\right)}\right) \\
&= \Delta\left(\prod_{i=1}^{n} \left(\Delta^{-1}\left(r_{index(i)}, \alpha_{index(i)}\right)\right)^{\left(\frac{R_i}{TV} - \frac{R_{i-1}}{TV}\right)}\right) \\
&= \Delta\left(\prod_{i=1}^{n} \left(\Delta^{-1}\left(r_{index(i)}, \alpha_{index(i)}\right)\right)^{\frac{V_{index(i)}}{TV}}\right) \\
&= \Delta\left(\prod_{i=1}^{n} \left(\Delta^{-1}\left(r_{index(i)}, \alpha_{index(i)}\right)\right)^{\left(1 + T\left(r_{index(i)}, \alpha_{index(i)}\right)\right) \Big/ \sum_{i=1}^{n}\left(1 + T\left(r_{index(i)}, \alpha_{index(i)}\right)\right)}\right) \\
&= \Delta\left(\prod_{i=1}^{n} \left(\Delta^{-1}\left(r_i, \alpha_i\right)\right)^{\left(1 + T(r_i, \alpha_i)\right) \big/ \sum_{i=1}^{n}\left(1 + T(r_i, \alpha_i)\right)}\right) \\
&= \text{2TLPG}\left((r_1, \alpha_1), (r_2, \alpha_2), \cdots, (r_n, \alpha_n)\right).
\end{aligned}$$

Furthermore, if $Sup\left((r_i, \alpha_i), (r_j, \alpha_j)\right) = k$ for all $i \neq j$ and $g(x) = x$, then we have

$$\text{2TLPOWG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)=\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}\left(r_i,\alpha_i\right)\right)^{1/n}\right)=\text{2TG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)$$

,
which indicates that when all of the supports are the same, the 2TLPOWG operator reduces to the 2-tuple geometric (2TG) operator [22].

Theorem 3.6. Suppose that $\left\{\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right\}$ $\left(r_i \in S, \alpha_i \in [-0.5, 0.5), i = 1,2,\cdots,n\right)$ is a collection of 2-tuples. Then, we have

$$\text{2TLPOWG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)\le \text{2TLPOWA}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right). \tag{29}$$

Theorem 3.6 shows that the values obtained with the 2TLPOWG operator are not larger than those obtained with the 2TLPOWA operator.

Theorem 3.7. Let $\left\{\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right\}$ $\left(r_i \in S, \alpha_i \in [-0.5, 0.5), i = 1,2,\cdots,n\right)$ be a collection of 2-tuples. Then, the following properties hold.

(1) Commutativity: If $\left\{\left(r_1',\alpha_1'\right),\left(r_2',\alpha_2'\right),\cdots,\left(r_n',\alpha_n'\right)\right\}$ is any permutation of $\left\{\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right\}$, then

$$\text{2TLPOWG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)=\text{2TLPOWG}\left(\left(r_1',\alpha_1'\right),\left(r_2',\alpha_2'\right),\cdots,\left(r_n',\alpha_n'\right)\right) \tag{30}$$

(2) Idempotency: If $\left(r_i,\alpha_i\right)=\left(r,\alpha\right)$ for all i, then

$$\text{2TLPOWG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)=\left(r,\alpha\right). \tag{31}$$

(3) Boundedness:

$$\min_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\}\le \text{2TLPOWG}\left(\left(r_1,\alpha_1\right),\left(r_2,\alpha_2\right),\cdots,\left(r_n,\alpha_n\right)\right)\le \max_{1\le i\le n}\left\{\left(r_i,\alpha_i\right)\right\} \tag{32}$$

2.7 Generalized 2-Tuple Linguistic Power Average (G2TLPA) Operators And Generalized 2-Tuple Linguistic Power Weighted Average (G2TLPWA) Operators

We now provide a generalization of the 2TLPA operator by combining it with the generalized mean operator [28] to obtain the generalized 2-tuple linguistic power average (G2TLPA) operator. The G2TLPA operator can be defined as follows:

Definition 3.4. For a collection of 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i \in S, \alpha_i \in [-0.5, 0.5), i = 1, 2, \cdots, n)$, a generalized 2-tuple linguistic power average (G2TLPA) operator is a mapping $H^n \to H$ such that

$$\mathrm{G2TLPA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) = \Delta\left(\left(\frac{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)\left(\Delta^{-1}(r_i,\alpha_i)\right)^\lambda}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}\right)^{1/\lambda}\right), \quad (33)$$

where $T(r_i,\alpha_i)$ satisfies Eq. (17).

From Definition 3.4, the weight vector $\frac{1+T(r_i,\alpha_i)}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}$ of the G2TLPA operator depends on the input arguments and allows the arguments being aggregated to support and reinforce one another.

In particular, if $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right) = k$ for all $i \neq j$, then

$$\mathrm{G2TLPA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) = \Delta\left(\left(\sum_{i=1}^{n}\frac{1}{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^\lambda\right)^{1/\lambda}\right),$$

which indicates that when all of the supports are the same, the G2TLPA operator reduces to the generalized 2-tuple linguistic average (G2TLA) operator [24].

If $\lambda = 1$, then the G2TLPA operator degenerates to the 2TLPA operator (Eq. (11)). If $Sup\left((r_i,\alpha_i),(r_j,\alpha_j)\right) = k$ for all $i \neq j$ and $\lambda = 1$, then the G2TLPA operator reduces to the 2-tuple average (2TA) operator [18].

Theorem 3.8. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i \in S, \alpha_i \in [-0.5, 0.5), i = 1, 2, \cdots, n)$ be a collection of 2-tuples and assume that $\lambda > 0$. Then,

$$\mathrm{2TLPG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \le \mathrm{G2TLPA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (34)$$

Proof. Because $\sum_{i=1}^{n}\frac{1+T(r_i,\alpha_i)}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)} = \frac{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)}{\sum_{i=1}^{n}\left(1+T(r_i,\alpha_i)\right)} = 1$, we have by Lemma 3.1:

$$\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{1+T(r_i,\alpha_i)/\sum_{i=1}^{n}(1+T(r_i,\alpha_i))} = \left(\prod_{i=1}^{n}\left(\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\lambda}\right)^{1+T(r_i,\alpha_i)/\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\right)^{1/\lambda}$$

$$\leq \left(\sum_{i=1}^{n}\left(\frac{1+T(r_i,\alpha_i)}{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\lambda}\right)\right)^{1/\lambda}.$$

By Theorem 2.2, we can therefore conclude that

$$\Delta\left(\prod_{i=1}^{n}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{1+T(r_i,\alpha_i)/\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\right) \leq \Delta\left(\left(\sum_{i=1}^{n}\left(\frac{1+T(r_i,\alpha_i)}{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\lambda}\right)\right)^{1/\lambda}\right)$$

,
which implies that

$$\mathrm{2TLPG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right) \leq \mathrm{G2TLPA}_{\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right).$$

According to Theorem 3.8, the values obtained with the 2TLPG operator are no larger than those obtained with the G2TLPA operator for any $\lambda > 0$.

Theorem 3.9. For a given collection of 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, the G2TLPA operator is monotonically increasing with respect to the parameter λ.

Proof. From the monotonicity of the GOWA operator [42], we obtain that $\left(\frac{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\lambda}}{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\right)^{1/\lambda}$ is monotonically increasing with respect to the parameter λ. By Theorem 2.2, we can therefore conclude that $\Delta\left(\left(\sum_{i=1}^{n}\left(\frac{1+T(r_i,\alpha_i)}{\sum_{i=1}^{n}(1+T(r_i,\alpha_i))}\left(\Delta^{-1}(r_i,\alpha_i)\right)^{\lambda}\right)\right)^{1/\lambda}\right)$ is monotonically increasing with respect to the parameter λ, which implies that the G2TLPA operator is monotonically increasing with respect to the parameter λ. o

Theorem 3.10. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $(r_i \in S, \alpha_i \in [-0.5,0.5), i=1,2,\cdots,n)$ be a collection of 2-tuples. Then, the following properties hold.

(1) Commutativity: If $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, then

$$\mathrm{G2TLPA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\mathrm{G2TLPA}_\lambda\left((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\right) \tag{35}$$

(2) Idempotency: If $(r_i,\alpha_i)=(r,\alpha)$ for all i, then

$$\mathrm{G2TLPA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=(r,\alpha) \tag{36}$$

(3) Boundedness:

$$\min_{1\le i\le n}\{(r_i,\alpha_i)\}\le \mathrm{G2TLPA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\le \max_{1\le i\le n}\{(r_i,\alpha_i)\} \tag{37}$$

In the G2TLPA operator, all the arguments that are aggregated are of equal importance. If we consider the weights of the arguments, then we can develop a generalized 2-tuple linguistic power weighted average (G2TLPWA) operator as follows:

Definition 3.5. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples. A generalized 2-tuple linguistic power weighted average (G2TLPWA) operator is defined as follows:

$$\mathrm{G2TLPWA}_{w,\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\left(\frac{\sum_{i=1}^{n}w_i\left(1+T'(r_i,\alpha_i)\right)\left(\Delta^{-1}(r_i,\alpha_i)\right)^\lambda}{\sum_{i=1}^{n}w_i\left(1+T'(r_i,\alpha_i)\right)}\right)^{1/\lambda}\right), \tag{38}$$

where $T'(r_i,\alpha_i)$ satisfies Eq. (24), $w_i\in[0,1]$ for $i=1,2,\ldots,n$, and $\sum_{i=1}^{n}w_i=1$.

We now consider some special cases obtained by using different choices for the parameters w and λ. If $\lambda=1$, then the G2TLPWA operator becomes the 2TLPWA operator (Eq. (13)). If $w=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then the G2TLPWA operator reduces to the G2TLPA operator (Eq. (33)). If $w=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$ and $\lambda=1$, then the G2TLPWA operator reduces to the 2TLPA operator (Eq. (11)).

Similar to the 2TLPWG operator, the G2TLPWA operator has the properties such as idempotency and boundedness, but commutativity property does not hold.

Theorem 3.11. For a given collection of 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$, the G2TLPWA operator is monotonically increasing with respect to the parameter λ.

Theorem 3.12. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples and assume that $\lambda>0$. Then,

$$\text{2TLPWG}_{w}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\leq \text{G2TLPWA}_{w,\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (39)$$

3.8 Generalized 2-Tuple Linguistic Power Ordered Weighted Average (G2TLPOWA) Operators

Based on the 2TLPOWA operator (Eq. (14)) and the generalized mean operator [28], we define a generalized 2-tuple linguistic power ordered weighted average (G2TLPOWA) operator as follows:

Definition 3.6. For a collection of 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$, a generalized 2-tuple linguistic power ordered weighted average (G2TLPOWA) operator is a mapping $H^n\rightarrow H$ such that

$$\text{G2TLPOWA}_{\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\Delta\left(\left(\sum_{i=1}^{n}\left(u_i\left(\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right), \quad (40)$$

where u_i satisfies Eq. (28) and $\left(r_{index(i)},\alpha_{index(i)}\right)$ is the *i*th largest argument of $\left(r_j,\alpha_j\right)$ $\left(j=1,2,\cdots,n\right)$.

In particular, if $\lambda=1$, then the G2TLPOWA operator reduces to the 2TLPOWA operator (Eq. (14)). If $g(x)=x$, then the G2TLPOWA operator reduces to the G2TLPA operator.

$$\begin{aligned}
&\mathrm{G2TLPOWA}_{\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\\
&=\Delta\left(\left(\sum_{i=1}^{n}\left(u_i\left(\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\\
&=\Delta\left(\left(\sum_{i=1}^{n}\left(\left(g\left(\frac{R_i}{TV}\right)-g\left(\frac{R_{i-1}}{TV}\right)\right)\left(\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\\
&=\Delta\left(\left(\sum_{i=1}^{n}\left(\left(\frac{R_i}{TV}-\frac{R_{i-1}}{TV}\right)\left(\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\\
&=\Delta\left(\left(\sum_{i=1}^{n}\left(\frac{V_{index(i)}}{TV}\left(\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\\
&=\Delta\left(\left(\sum_{i=1}^{n}\left(\frac{1+T\left(r_{index(i)},\alpha_{index(i)}\right)}{\sum_{i=1}^{n}\left(1+T\left(r_{index(i)},\alpha_{index(i)}\right)\right)}\left(\Delta^{-1}\left(r_{index(i)},\alpha_{index(i)}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\\
&=\Delta\left(\left(\sum_{i=1}^{n}\left(\frac{1+T\left(r_i,\alpha_i\right)}{\sum_{i=1}^{n}\left(1+T\left(r_i,\alpha_i\right)\right)}\left(\Delta^{-1}\left(r_i,\alpha_i\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\\
&=\mathrm{G2TLPA}_{\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right).
\end{aligned}$$

Theorem 3.13. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples. Then, the following properties hold.

(1) Commutativity: If $\{(r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\}$ is any permutation of $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$, then

$$\mathrm{G2TLPOWA}_{\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=\mathrm{G2TLPOWA}_{\lambda}\left((r_1',\alpha_1'),(r_2',\alpha_2'),\cdots,(r_n',\alpha_n')\right). \quad (41)$$

(2) Idempotency: If $(r_i,\alpha_i)=(r,\alpha)$ for all i, then

$$\mathrm{G2TLPOWA}_{\lambda}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)=(r,\alpha). \quad (42)$$

(3) Boundedness:

$$\min_{1\le i\le n}\{(r_i,\alpha_i)\}\le \mathrm{G2TLPOWA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\le \max_{1\le i\le n}\{(r_i,\alpha_i)\}. \quad (43)$$

Theorem 3.14. For the given 2-tuples $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$, the G2TLPOWA operator is monotonically increasing with respect to the parameter λ.

Theorem 3.15. Let $\{(r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\}$ $\left(r_i\in S,\alpha_i\in[-0.5,0.5),i=1,2,\cdots,n\right)$ be a collection of 2-tuples and assume that $\lambda>0$. Then,

$$\mathrm{2TLPOWG}\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right)\le \mathrm{G2TLPOWA}_\lambda\left((r_1,\alpha_1),(r_2,\alpha_2),\cdots,(r_n,\alpha_n)\right). \quad (44)$$

According to Theorem 3.15, the values obtained with the 2TLPOWG operator are no larger than those obtained with the G2TLPOWA operator for any $\lambda>0$.

Remark 3.1. In this section, we have proposed six aggregation operators for aggregating 2-tuple linguistic information, which can be divided into two categories: 2-tuple linguistic power aggregation operators and 2-tuple linguistic power ordered weighted aggregation operators. The former, including the 2TLPG, 2TLPWG, G2TLPA, and G2TLPWA, emphasize the importance of each argument, i.e., the closer an argument is to the middle one(s), the higher its weight. The latter, including the 2TLPOWG and G2TLPOWA operators, weight the importance of each argument according to its ordered position, i.e., the closer the ordered position of the argument is to the middle one(s), the higher its weight.

In real-life situations, the arguments sometimes take the form of a collection of 2-tuples provided by different individuals. In recent years, many aggregation operators have been developed for aggregating 2-tuple linguistic information, such as the 2-tuple arithmetic mean operator [18,20], 2-tuple weighted averaging operator [18], 2-tuple OWA operator [18], TWGA operator [21], TOWGA operator [21], THGA operator [21], TAA operator [18], TWA operator [18], TOWA operator [18], ET-WA operator [18], TOWG operator [22], ET-WG operator [23], ET-OWG operator [23], G-2TWA operator [24], G-2TOWA operator [24], and IG-2TOWA operator [24]. However, these 2-tuple linguistic aggregation operators cannot capture the sophisticated nuances that the decision makers wish to reflect in the aggregated value, i.e., these aggregation operators cannot take into account the relationships between the arguments provided by different individuals. However, the new 2-tuple linguistic power aggregation operators proposed in this paper can not only incorporate the relationships between the input arguments by allowing the values being aggregated to support and reinforce one another but also measure the similarity degrees of the arguments and thereby reduce the influence of unduly high or low arguments on the decision result by using the support measure to assign them lower weights. In the process of group decision making, some individuals may provide unduly high or low evaluation values to their preferred or dispreferred objects. Compared to the previously proposed 2-tuple linguistic aggregation operators, the new operators have the advantage that the associated weights are determined using the support measure. If the preference value provided by a decision maker is more similar (or closer) to the values provided by the other decision makers, then it receives a higher weight; the operators can therefore reduce the influence of these unduly high or low arguments of the individual information, thereby providing a more reliable decision result.

3. Approaches to Multiple Attribute Group Decision Making With 2-Tuple Linguistic Information

In this section, we utilize the proposed 2-tuple power aggregation operators to develop some approaches to multiple attribute group decision making with 2-tuple linguistic information.

The multiple attribute group decision making problem with 2-tuple linguistic information can be formulated as follows:

Let $X=\{x_1,x_2,\cdots,x_m\}$ be a set of m alternatives, and let $C=\{c_1,c_2,\ldots,c_n\}$ be a collection of n attributes, whose weight vector is $w=(w_1,w_2,\cdots,w_n)^T$, with $w_i\in[0,1]$, $i=1,2,\cdots,n$, and $\sum_{i=1}^{n}w_i=1$, and let $D=\{d_1,d_2,\cdots,d_l\}$ be a set of l decision makers, whose weight vector is $\omega=(\omega_1,\omega_2,\cdots,\omega_l)^T$ with $\omega_k\in[0,1]$, $k=1,2,\cdots,l$, and $\sum_{k=1}^{l}\omega_k=1$. Each decision maker provides his/hier own linguistic decision matrix $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$ $(k=1,2,\cdots,l)$, where $r_{ij}^{(k)}\in S$ is a preference value, which takes the form of linguistic variable, given by the decision maker $d_k\in D$, for the alternative $x_i\in X$ with respect to the attribute $c_j\in C$.
In the following, we utilize the G2TLPWA (or 2TLPWG) operator to develop an approach to multi-attribute group decision making in a 2-tuple linguistic environment. The algorithm involves the following steps.

Approach I

Step 1. Transform the linguistic decision matrix $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$ $(k=1,2,\cdots,l)$ into 2-tuple linguistic decision matrix $\overline{R}^{(k)}=\left(\left(r_{ij}^{(k)},0\right)\right)_{m\times n}$ $(k=1,2,\cdots,l)$.
Step 2. Calculate the supports,

$$Sup\left(\left(r_{ij}^{(k)},0\right),\left(r_{ij}^{(t)},0\right)\right)=1-d\left(\left(r_{ij}^{(k)},0\right),\left(r_{ij}^{(t)},0\right)\right),\quad k,t=1,2,\cdots,l,\quad i=1,2,\cdots,m,\quad j=1,2,\cdots,n, \tag{45}$$

which satisfy support conditions (1)-(3) in Definition 3.1. Without loss of generality, we assume that $d\left(\left(r_{ij}^{(k)},0\right),\left(r_{ij}^{(t)},0\right)\right)$ is the distance between $\left(r_{ij}^{(k)},0\right)$ and $\left(r_{ij}^{(t)},0\right)$ given in [33]:

$$d\left(\left(r_{ij}^{(k)},0\right),\left(r_{ij}^{(t)},0\right)\right)=\frac{1}{g}\left|\Delta^{-1}\left(r_{ij}^{(k)},0\right)-\Delta^{-1}\left(r_{ij}^{(k)},0\right)\right|. \tag{46}$$

Step 3. Utilize the weights ω_k $(k=1,2,\cdots,l)$ of the decision makers, d_k $(k=1,2,\cdots,l)$, to calculate the weighted support $T'\left(r_{ij}^{(k)},0\right)$ of 2-tuple $\left(r_{ij}^{(k)},0\right)$ by the other 2-tuples, $\left(r_{ij}^{(t)},0\right)$ ($t=1,2,\cdots,l$, and $t\neq k$):

$$T'\left(r_{ij}^{(k)},0\right)=\sum_{\substack{t=1\\t\neq k}}^{l}\omega_t Sup\left(\left(r_{ij}^{(k)},0\right),\left(r_{ij}^{(t)},0\right)\right) \tag{47}$$

Then, utilize the weights ω_k $(k=1,2,\cdots,l)$ of the decision makers, d_k $(k=1,2,\cdots,l)$, to calculate the weights $\xi_{ij}^{(k)}$ $(k=1,2,\cdots,l)$ associated with 2-tuples $\left(r_{ij}^{(k)},0\right)$ $(k=1,2,\cdots,l)$:

$$\xi_{ij}^{(k)}=\frac{\omega_k\left(1+T'\left(r_{ij}^{(k)},0\right)\right)}{\sum_{k=1}^{l}\omega_k\left(1+T'\left(r_{ij}^{(k)},0\right)\right)},\quad k=1,2,\cdots,l, \tag{48}$$

where $\xi_{ij}^{(k)}\geq 0$, $k=1,2,\cdots,l$, and $\sum_{k=1}^{l}\xi_{ij}^{(k)}=1$.

Step 4. Use the G2TLPWA operator (Eq. (38)),

$$\begin{aligned}\overline{r}_{ij}&=\left(r_{ij},\alpha_{ij}\right)=\mathrm{G2TLPWA}_{w,\lambda}\left(\left(r_{ij}^{(1)},0\right),\left(r_{ij}^{(2)},0\right),\ldots,\left(r_{ij}^{(l)},0\right)\right)\\&=\Delta\left(\left(\sum_{k=1}^{l}\left(\xi_{ij}^{(k)}\left(\Delta^{-1}\left(r_{ij}^{(k)},0\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\end{aligned} \tag{49}$$

or the 2TLPWG operator (Eq. (23)),

$$\overline{r}_{ij}=\left(r_{ij},\alpha_{ij}\right)=\mathrm{2TLPWG}_{w}\left(\left(r_{ij}^{(1)},0\right),\left(r_{ij}^{(2)},0\right),\ldots,\left(r_{ij}^{(l)},0\right)\right)=\Delta\left(\prod_{k=1}^{l}\left(\Delta^{-1}\left(r_{ij}^{(k)},0\right)\right)^{\xi_{ij}^{(k)}}\right) \tag{50}$$

to aggregate all of the individual 2-tuple linguistic decision matrices, $\overline{R}^{(k)}=\left(\left(r_{ij}^{(k)},0\right)\right)_{m\times n}$ ($k=1,2,\cdots,l$), into the collective 2-tuple linguistic decision matrix, $\overline{R}=\left(\overline{r}_{ij}\right)_{m\times n}=\left(\left(r_{ij},\alpha_{ij}\right)\right)_{m\times n}$.

Step 5. Calculate the supports:

$$Sup\left(\left(r_{ij},\alpha_{ij}\right),\left(r_{ip},\alpha_{ip}\right)\right)=1-d\left(\left(r_{ij},\alpha_{ij}\right),\left(r_{ip},\alpha_{ip}\right)\right),\ i=1,2,\cdots,m,\ j,p=1,2,\cdots,n, \tag{51}$$

which satisfy support conditions (1)-(3) in Definition 3.1. Here, we assume that $d\left(\left(r_{ij},\alpha_{ij}\right),\left(r_{ip},\alpha_{ip}\right)\right)$ is the distance between $\left(r_{ij},\alpha_{ij}\right)$ and $\left(r_{ip},\alpha_{ip}\right)$ given in [33]:

$$d\left(\left(r_{ij},\alpha_{ij}\right),\left(r_{ip},\alpha_{ip}\right)\right)=\frac{1}{g}\left|\Delta^{-1}\left(r_{ij},\alpha_{ij}\right)-\Delta^{-1}\left(r_{ip},\alpha_{ip}\right)\right|. \tag{52}$$

Step 6. Use the weights $w_j\ \left(j=1,2,\cdots,n\right)$ of attributes $c_j\ \left(j=1,2,\cdots,n\right)$ to calculate the weighted support $T'\left(r_{ij},\alpha_{ij}\right)$ of the 2-tuple $\left(r_{ij},\alpha_{ij}\right)$ by the other 2-tuples, $\left(r_{ip},\alpha_{ip}\right)$ ($p=1,2,\cdots,n$, and $p\neq j$):

$$T'\left(r_{ij},\alpha_{ij}\right)=\sum_{\substack{p=1\\p\neq j}}^{n}w_p Sup\left(\left(r_{ij},\alpha_{ij}\right),\left(r_{ip},\alpha_{ip}\right)\right). \tag{53}$$

Then, utilize the weights $w_j\ \left(j=1,2,\cdots,n\right)$ of attributes $c_j\ \left(j=1,2,\cdots,n\right)$ to calculate the weights $\eta_{ij}\ \left(j=1,2,\cdots,n\right)$ associated with 2-tuple $\left(r_{ij},\alpha_{ij}\right)\ \left(j=1,2,\cdots,n\right)$:

$$\eta_{ij}=\frac{w_j\left(1+T'\left(r_{ij},\alpha_{ij}\right)\right)}{\sum_{j=1}^{n}w_j\left(1+T'\left(r_{ij},\alpha_{ij}\right)\right)},\quad j=1,2,\cdots,n, \tag{54}$$

where $\eta_{ij}\geq 0$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n}\eta_{ij}=1$.

Step 7. Utilize the G2TLPWA operator (Eq. (38)),

$$\overline{r}_i=\left(r_i,\alpha_i\right)=\mathrm{G2TLPWA}\left(\left(r_{i1},\alpha_{i1}\right),\left(r_{i2},\alpha_{i2}\right),\ldots,\left(r_{in},\alpha_{in}\right)\right)=\Delta\left(\left(\sum_{j=1}^{n}\left(\eta_{ij}\left(\Delta^{-1}\left(r_{ij},\alpha_{ij}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right) \tag{55}$$

or the 2TLPWG operator (Eq. (23)),

$$\overline{r}_i=\left(r_i,\alpha_i\right)=\mathrm{2TLPWG}\left(\left(r_{i1},\alpha_{i1}\right),\left(r_{i2},\alpha_{i2}\right),\ldots,\left(r_{in},\alpha_{in}\right)\right)=\Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{ij},\alpha_{ij}\right)\right)^{\eta_{ij}}\right) \tag{56}$$

to aggregate all of the preference values $\overline{r}_{ij}$ ($j=1,2,\cdots,n$) in the *i*th line of $\overline{R}$, and then derive the collective overall preference value, $\overline{r}_i=\left(r_i,\alpha_i\right)\ \left(i=1,2,\cdots,m\right)$, of alternative x_i $\left(i=1,2,\cdots,m\right)$.

Step 8. Rank the $\overline{r}_i=\left(r_i,\alpha_i\right)\ \left(i=1,2,\cdots,m\right)$ in descending order using Definition 2.3.

Step 9. Rank all of the alternatives, $x_i\ \left(i=1,2,\cdots,m\right)$, and then select the best one(s) in accordance with the collective overall preference values, $\overline{r}_i=\left(r_i,\alpha_i\right)\ \left(i=1,2,\cdots,m\right)$.

Step 10. End.

If the information regarding the weights of the decision makers and attributes is unknown, then we utilize the G2TLPOWA (or 2TLPOWG) operator to develop an alternative approach to the MAGDM problem with 2-tuple linguistic information, which is described below.

Approach II

Step 1. Transform the linguistic decision matrix $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$ $\left(k=1,2,\cdots,l\right)$ into 2-tuple linguistic decision matrix $\bar{R}^{(k)}=\left(\left(r_{ij}^{(k)},0\right)\right)_{m\times n}$ $\left(k=1,2,\cdots,l\right)$.

Step 2. Calculate the supports:

$$Sup\left(\left(r_{ij}^{(index(k))},0\right),\left(r_{ij}^{(index(t))},0\right)\right)=1-d\left(\left(r_{ij}^{(index(k))},0\right),\left(r_{ij}^{(index(t))},0\right)\right),\ k,t=1,2,\cdots,l,$$
$$i=1,2,\cdots,m,\ j=1,2,\cdots,n, \qquad (57)$$

which satisfy the support conditions of Eq. (28) in Definition 3.3. We assume that $d\left(\left(r_{ij}^{(index(k))},0\right),\left(r_{ij}^{(index(t))},0\right)\right)$ is the distance between $\left(r_{ij}^{|index(k)|},0\right)$ and $\left(r_{ij}^{(index(t))},0\right)$ given in [33]:

$$d\left(\left(r_{ij}^{(index(k))},0\right),\left(r_{ij}^{(index(t))},0\right)\right)=\frac{1}{g}\left|\Delta^{-1}\left(r_{ij}^{(index(k))},0\right)-\Delta^{-1}\left(r_{ij}^{(index(t))},0\right)\right|. \qquad (58)$$

Step 3. Calculate the support $T\left(r_{ij}^{(index(k))},0\right)$ of the *k*th largest 2-tuple $\left(r_{ij}^{(index(k))},0\right)$ by the other 2-tuples $\left(r_{ij}^{(index(t))},0\right)$ ($t=1,2,\cdots,l$, and $t\neq k$):

$$T\left(r_{ij}^{(index(k))},0\right)=\sum_{\substack{t=1\\t\neq k}}^{l}Sup\left(\left(r_{ij}^{(index(k))},0\right),\left(r_{ij}^{(index(t))},0\right)\right) \qquad (59)$$

and then utilize Eq. (28) to calculate the weight $u_{ij}^{(k)}$ $\left(k=1,2,\cdots,l\right)$ associated with the *k*th largest 2-tuple $\left(r_{ij}^{index(k)},0\right)$ $\left(k=1,2,\cdots,l\right)$, where

$$u_{ij}^{(k)}=g\left(\frac{B_{ij}^{(k)}}{TV_{ij}}\right)-g\left(\frac{B_{ij}^{(k-1)}}{TV_{ij}}\right),\ B_{ij}^{(k)}=\sum_{h=1}^{k}V_{ij}^{index(h)},\ TV_{ij}=\sum_{h=1}^{l}V_{ij}^{index(h)},\ V_{ij}^{index(h)}=1+T\left(r_{ij}^{index(h)},0\right), \qquad (60)$$

$u_{ij}^{(k)} \ge 0$, $k = 1, 2, \cdots, l$, $\sum_{k=1}^{l} u_{ij}^{(k)} = 1$, and g is the BUM function, as described in Definition 3.3.

Step 4. Utilize the G2TLPOWA operator (Eq. (40)),

$$\overline{r}_{ij} = \left(r_{ij}, \alpha_{ij}\right) = \text{G2TLPOWA}\left(\left(r_{ij}^{(1)}, 0\right), \left(r_{ij}^{(2)}, 0\right), \ldots, \left(r_{ij}^{(l)}, 0\right)\right) = \Delta\left(\left(\sum_{k=1}^{l}\left(u_{ij}^{(k)}\left(\Delta^{-1}\left(r_{ij}^{index(k)}, 0\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right) \quad (61)$$

or the 2TLPOWG operator (Eq. (27)),

$$\overline{r}_{ij} = \left(r_{ij}, \alpha_{ij}\right) = \text{2TLPOWG}\left(\left(r_{ij}^{(1)}, 0\right), \left(r_{ij}^{(2)}, 0\right), \ldots, \left(r_{ij}^{(l)}, 0\right)\right) = \Delta\left(\prod_{k=1}^{l}\left(\Delta^{-1}\left(r_{ij}^{index(k)}, 0\right)\right)^{u_{ij}^{(k)}}\right) \quad (62)$$

to aggregate all of the 2-tuple linguistic decision matrices, $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{m \times n}$ ($k = 1, 2, \cdots, l$), into the collective 2-tuple linguistic decision matrix, $\overline{R} = \left(\overline{r}_{ij}\right)_{m \times n} = \left(\left(r_{ij}, \alpha_{ij}\right)\right)_{m \times n}$.

Step 5. Calculate the supports:

$$Sup\left(\left(r_{iindex(j)}, \alpha_{iindex(j)}\right), \left(r_{iindex(p)}, \alpha_{iindex(p)}\right)\right) = 1 - d\left(\left(r_{iindex(j)}, \alpha_{iindex(j)}\right), \left(r_{iindex(p)}, \alpha_{iindex(p)}\right)\right),$$
$$i = 1, 2, \cdots, m, \; j, p = 1, 2, \cdots, n, \quad (63)$$

which satisfy the support conditions of Eq. (28) in Definition 3.3. Here, $\left(r_{iindex(j)}, \alpha_{iindex(j)}\right)$ is the *j*th largest 2-tuple among all of the 2-tuples $\left(r_{iq}, \alpha_{iq}\right)\left(q = 1, 2, \cdots, n\right)$, and $d\left(\left(r_{iindex(j)}, \alpha_{iindex(j)}\right), \left(r_{iindex(p)}, \alpha_{iindex(p)}\right)\right)$ is the distance between $\left(r_{iindex(j)}, \alpha_{iindex(j)}\right)$ and $\left(r_{iindex(p)}, \alpha_{iindex(p)}\right)$ given in [33]:

$$d\left(\left(r_{iindex(j)}, \alpha_{iindex(j)}\right), \left(r_{iindex(p)}, \alpha_{iindex(p)}\right)\right) = \frac{1}{g}\left|\Delta^{-1}\left(r_{iindex(j)}, \alpha_{iindex(j)}\right) - \Delta^{-1}\left(r_{iindex(p)}, \alpha_{iindex(p)}\right)\right|. \quad (64)$$

Step 6. Calculate the support $T\left(r_{iindex(j)}, \alpha_{iindex(j)}\right)$ of the *j*th largest 2-tuple, $\left(r_{iindex(j)}, \alpha_{iindex(j)}\right)$, by the other 2-tuples, $\left(r_{iindex(p)}, \alpha_{iindex(p)}\right)$ ($p = 1, 2, \cdots, n$, and $p \ne j$):

$$T\left(r_{iindex(j)}, \alpha_{iindex(j)}\right) = \sum_{\substack{p=1 \\ p \ne j}}^{n} Sup\left(\left(r_{iindex(j)}, \alpha_{iindex(j)}\right), \left(r_{iindex(p)}, \alpha_{iindex(p)}\right)\right). \quad (65)$$

Then, utilize Eq. (28) to calculate the weight u_{ij} $(j=1,2,\cdots,n)$ associated with the *j*th largest 2-tuple, $\left(r_{iindex(j)},\alpha_{iindex(j)}\right)$ $(j=1,2,\cdots,n)$, where

$$u_{ij}=g\left(\frac{B_{ij}}{TV_i}\right)-g\left(\frac{B_{i(j-1)}}{TV_i}\right),\quad B_{ij}=\sum_{h=1}^{j}V_{iindex(h)},\quad TV_i=\sum_{h=1}^{n}V_{iindex(h)},$$
$$V_{iindex(h)}=1+T\left(\left(r_{iindex(h)},\alpha_{iindex(h)}\right)\right),\tag{66}$$

$u_{ij}\geq 0$, $j=1,2,\cdots,n$, $\sum_{j=1}^{n}u_{ij}=1$, and g is the BUM function, as described in Definition 3.3.

Step 7. Use the G2TLPOWA operator (Eq. (40)),

$$\bar{r}_i=(r_i,\alpha_i)=\text{G2TLPOWA}\left((r_{i1},\alpha_{i1}),(r_{i2},\alpha_{i2}),\ldots,(r_{in},\alpha_{in})\right)=\Delta\left(\left(\sum_{j=1}^{n}\left(u_{ij}\left(\Delta^{-1}\left(r_{iindex(j)},\alpha_{iindex(j)}\right)\right)^{\lambda}\right)\right)^{1/\lambda}\right)\tag{67}$$

or the 2TLPOWG operator (Eq. (27)),

$$\bar{r}_i=(r_i,\alpha_i)=\text{2TLPOWG}\left((r_{i1},\alpha_{i1}),(r_{i2},\alpha_{i2}),\ldots,(r_{in},\alpha_{in})\right)=\Delta\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(r_{iindex(j)},\alpha_{iindex(j)}\right)\right)^{u_{ij}}\right)\tag{68}$$

to aggregate all of the preference values (r_{ij},α_{ij}) ($j=1,2,\cdots,n$) in the *i*th line of $\bar{R}$, and then derive the collective overall preference value, $\bar{r}_i=(r_i,\alpha_i)$ $(i=1,2,\cdots,m)$, of alternative x_i $(i=1,2,\cdots,m)$.

Step 8. Rank the $\bar{r}_i=(r_i,\alpha_i)$ $(i=1,2,\cdots,m)$ in descending order using Definition 2.3.

Step 9. Rank all of the alternatives, x_i $(i=1,2,\cdots,m)$, and then select the best one(s) in accordance with the collective overall preference values, $\bar{r}_i=(r_i,\alpha_i)$ $(i=1,2,\cdots,m)$.

Step 10. End.

Remark 4.1. Approach I is designed for situations where the weights of the decision makers and attributes can be predefined and utilizes the G2TLPWA (or 2TLPWG) operator to aggregate all of the individual 2-tuple linguistic decision matrices into the collective 2-tuple linguistic decision matrix. If we emphasize the individual influence, then the 2TLPWG operator based on the geometric aggregating tool is available; if we emphasize the group's influence, then the G2TLPWA operator based on the arithmetic aggregating tool is available. Approach II is designed for situations where the information regarding the weights of the decision makers and attributes is unknown and utilizes the G2TLPOWA (or 2TLPOWG) operator to aggregate all of the individual 2-tuple linguistic decision matrices into the collective 2-tuple linguistic decision matrix. If we emphasize the individual influence, then the 2TLPOWG operator based on the geometric aggregating tool is available; if we emphasize the group's influence, then the G2TLPOWA operator based on the arithmetic aggregating tool is available. Both approaches are quite suitable for multiple attribute group decision making in 2-tuple linguistic environments. In the process of group decision making, some individuals may assign unduly high or low preferences to their preferred or dispreferred objects. The proposed approaches can reduce the influence of these unduly high or low arguments on the decision result by using the support measure to assign lower weights to them, making the decision more reliable.

4. ILLUSTRATIVE EXAMPLES

In this subsection, let us consider a numerical example adapted from Herrera et al. [43], and Herrera and Martínez [34].

Example 5.1. Suppose that an investment company wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money: (1) x_1 is a car industry; (2) x_2 is a food company; (3) x_3 is a computer company; and (4) x_4 is an arms industry. The investment company must make a decision according to the following four attributes: (1) c_1 is the risk analysis; (2) c_2 is the growth analysis; (3) c_3 is the social-political impact analysis; and (4) c_4 is the environmental impact analysis. The weight vector of attributes c_j $(j=1,2,3,4)$ is $w=(0.3,0.25,0.25,0.2)^T$. The four possible alternatives x_i $(i=1,2,3,4)$ are to be evaluated using the linguistic term set

$$S=\left\{\begin{array}{l} s_0=\text{extremely poor},\ s_1=\text{very poor},\ s_2=\text{poor},\ s_3=\text{slightly poor},\ s_4=\text{fair}, \\ s_5=\text{slightly good},\ s_6=\text{good},\ s_7=\text{very good},\ s_8=\text{extremely good} \end{array}\right\}$$

by three decision makers d_k $(k=1,2,3)$ (suppose that the weight vector of three decision makers is $\omega=(0.2,0.5,0.3)^T$) under the above four attributes, and construct, respectively, the linguistic decision matrices $R^{(k)}=\left(r_{ij}^{(k)}\right)_{4\times 4}$ $(k=1,2,3)$ as shown in Tables 1-3.

Table 1. Linguistic decision matrix $R^{(1)}$ provided by d_1

1	c_1	c_2	c_3	c_4
x_1	s_4	s_3	s_1	s_5
x_2	s_3	s_6	s_5	s_8
x_3	s_3	s_2	s_7	s_5
x_4	s_8	s_1	s_3	s_6

Table 2. Linguistic decision matrix $R^{(2)}$ provided by d_2

2	c_1	c_2	c_3	c_4
x_1	s_5	s_2	s_7	s_3
x_2	s_7	s_4	s_8	s_6
x_3	s_7	s_8	s_5	s_6
x_4	s_8	s_6	s_5	s_3

Table 3. Linguistic decision matrix $R^{(3)}$ provided by d_3

3	c_1	c_2	c_3	c_4
x_1	s_5	s_1	s_2	s_8
x_2	s_7	s_8	s_6	s_5
x_3	s_5	s_6	s_3	s_4
x_4	s_6	s_8	s_5	s_7

Assume that the weights of the decision makers and attributes are known. We use Approach I to find the decision result.

Step 1. Transform the linguistic decision matrices $R^{(k)} = \left(r_{ij}^{(k)}\right)_{4\times4}$ $(k = 1,2,3)$ given in Tables 1-3 into 2-tuple linguistic decision matrices $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{4\times4}$ $(k = 1,2,3)$ which are given in Tables 4-6.

Table 4. 2-Tuple linguistic decision matrix $\overline{R}^{(1)}$

4	c_1	c_2	c_3	c_4
x_1	$(s_4, 0)$	$(s_3, 0)$	$(s_1, 0)$	$(s_5, 0)$
x_2	$(s_3, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_8, 0)$
x_3	$(s_3, 0)$	$(s_2, 0)$	$(s_7, 0)$	$(s_5, 0)$
x_4	$(s_8, 0)$	$(s_1, 0)$	$(s_3, 0)$	$(s_6, 0)$

Table 5. 2-tuple linguistic decision matrix $\overline{R}^{(2)}$

5	c_1	c_2	c_3	c_4
x_1	$(s_5, 0)$	$(s_2, 0)$	$(s_7, 0)$	$(s_3, 0)$
x_2	$(s_7, 0)$	$(s_4, 0)$	$(s_8, 0)$	$(s_6, 0)$
x_3	$(s_7, 0)$	$(s_8, 0)$	$(s_5, 0)$	$(s_6, 0)$
x_4	$(s_8, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_3, 0)$

Table 6. 2-tuple linguistic decision matrix $\overline{R}^{(3)}$

6	c_1	c_2	c_3	c_4
x_1	$(s_5,0)$	$(s_1,0)$	$(s_2,0)$	$(s_8,0)$
x_2	$(s_7,0)$	$(s_8,0)$	$(s_6,0)$	$(s_5,0)$
x_3	$(s_5,0)$	$(s_6,0)$	$(s_3,0)$	$(s_4,0)$
x_4	$(s_6,0)$	$(s_8,0)$	$(s_5,0)$	$(s_7,0)$

Step 2. Use Eq. (45) to calculate the supports, $Sup\left(\left(r_{ij}^{(k)},0\right),\left(r_{ij}^{(t)},0\right)\right)$ ($i,j=1,2,3,4$, $k,t=1,2,3$, $k\neq t$). For simplicity, we denote $\left(Sup\left(\left(r_{ij}^{(k)},0\right),\left(r_{ij}^{(t)},0\right)\right)\right)_{4\times4}$ by Sup^{kt}, which refers to the supports between $\overline{R}^{(k)}$ and $\overline{R}^{(t)}$, in the following:

,

$$Sup^{13}=Sup^{31}=\begin{bmatrix}0.8750 & 0.7500 & 0.8750 & 0.6250\\ 0.5000 & 0.7500 & 0.8750 & 0.6250\\ 0.7500 & 0.5000 & 0.5000 & 0.8750\\ 0.7500 & 0.1250 & 0.7500 & 0.8750\end{bmatrix},$$

$$Sup^{12}=Sup^{21}=\begin{bmatrix}0.8750 & 0.8750 & 0.2500 & 0.7500\\ 0.5000 & 0.7500 & 0.6250 & 0.7500\\ 0.5000 & 0.2500 & 0.7500 & 0.8750\\ 1.0000 & 0.3750 & 0.7500 & 0.6250\end{bmatrix}$$

$$Sup^{23}=Sup^{32}=\begin{bmatrix}1.0000 & 0.8750 & 0.3750 & 0.3750\\ 1.0000 & 0.5000 & 0.7500 & 0.8750\\ 0.7500 & 0.7500 & 0.7500 & 0.7500\\ 0.7500 & 0.7500 & 1.0000 & 0.5000\end{bmatrix}.$$

Step 3. Use Eq. (47) to calculate the weighted support $T'\left(r_{ij}^{(k)},0\right)$ of 2-tuple $\left(r_{ij}^{(k)},0\right)$ by the other 2-tuples, $\left(r_{ij}^{(t)},0\right)$ ($t=1,2,3$, and $t\neq k$). We denote $\left(T'\left(r_{ij}^{(k)},0\right)\right)_{4\times4}$ by T_k' ($k=1,2,3$) in the following equations:

$$T_1'=\begin{bmatrix}0.7000 & 0.6625 & 0.3875 & 0.5625\\ 0.4000 & 0.6000 & 0.5750 & 0.5625\\ 0.4750 & 0.2750 & 0.5250 & 0.7000\\ 0.7250 & 0.2250 & 0.6000 & 0.5750\end{bmatrix},\quad T_2'=\begin{bmatrix}0.4750 & 0.4375 & 0.1625 & 0.2625\\ 0.4000 & 0.3000 & 0.3500 & 0.4125\\ 0.3250 & 0.2750 & 0.3750 & 0.4000\\ 0.4250 & 0.3000 & 0.4500 & 0.2750\end{bmatrix},$$

$$T_3' = \begin{bmatrix} 0.6750 & 0.5875 & 0.3625 & 0.3125 \\ 0.6000 & 0.4000 & 0.5500 & 0.5625 \\ 0.5250 & 0.4750 & 0.4750 & 0.5500 \\ 0.5250 & 0.4000 & 0.6500 & 0.4250 \end{bmatrix}.$$

Use Eq. (48) to calculate the weights $\xi_{ij}^{(k)}$ $(i, j = 1,2,3,4,\ k = 1,2,3)$ of 2-tuple $\left(r_{ij}^{(k)}, 0\right)$ $(i, j = 1,2,3,4,\ k = 1,2,3)$. We denote $\left(\xi_{ij}^{(k)}\right)_{4\times4}$ by V_k ($k = 1,2,3$) in the following:

$$V_1 = \begin{bmatrix} 0.2152 & 0.2177 & 0.2189 & 0.2336 \\ 0.1918 & 0.2302 & 0.2165 & 0.2101 \\ 0.2085 & 0.1910 & 0.2125 & 0.2259 \\ 0.2277 & 0.1863 & 0.2078 & 0.2283 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.4668 & 0.4705 & 0.4586 & 0.4720 \\ 0.4795 & 0.4676 & 0.4639 & 0.4748 \\ 0.4682 & 0.4775 & 0.4791 & 0.4651 \\ 0.4703 & 0.4943 & 0.4708 & 0.4620 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} 0.3180 & 0.3118 & 0.3225 & 0.2944 \\ 0.3288 & 0.3022 & 0.3196 & 0.3151 \\ 0.3233 & 0.3315 & 0.3084 & 0.3090 \\ 0.3020 & 0.3194 & 0.3214 & 0.3098 \end{bmatrix}.$$

Step 4. Let $\lambda = 2$. Use the G2TLPWA operator (Eq. (49)) to aggregate all of the individual 2-tuple linguistic decision matrices $\bar{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{4\times4}$ ($k = 1,2,3$) into the collective 2-tuple linguistic decision matrix, $\bar{R} = \left(\bar{r}_{ij}\right)_{4\times4} = \left(\left(r_{ij}, \alpha_{ij}\right)\right)_{4\times4}$ (see Table 7).

Table 7. The collective 2-tuple linguistic decision matrix $\bar{R}$

7	c_1	c_2	c_3	c_4
x_1	$(s_5, -0.1976)$	$(s_2, 0.0379)$	$(s_5, -0.1031)$	$(s_5, 0.3787)$
x_2	$(s_6, 0.4287)$	$(s_6, -0.0748)$	$(s_7, -0.1730)$	$(s_6, 0.1981)$
x_3	$(s_6, -0.2641)$	$(s_7, -0.4229)$	$(s_5, 0.0167)$	$(s_5, 0.2283)$
x_4	$(s_7, 0.4528)$	$(s_6, 0.1986)$	$(s_5, -0.3443)$	$(s_5, 0.2492)$

Step 5. Use Eq. (51) to calculate the supports, $Sup\left(\left(r_{ij},\alpha_{ij}\right),\left(r_{ip},\alpha_{ip}\right)\right)$ ($i=1,2,3,4$, $j,p=1,2,3,4$, $j\neq p$). For simplicity, we denote $\left(Sup\left(\left(r_{ij},\alpha_{ij}\right),\left(r_{ip},\alpha_{ip}\right)\right)\right)_{4\times1}$ by Sup_{jp} , which refers to the supports between the *j*th and *p*th rows of $\overline{R}$, in the following:

$$Sup_{12}=Sup_{21}=\begin{bmatrix}0.6544\\0.9371\\0.8949\\0.8432\end{bmatrix},\quad Sup_{13}=Sup_{31}=\begin{bmatrix}0.9882\\0.9502\\0.9101\\0.6504\end{bmatrix},\quad Sup_{14}=Sup_{41}=\begin{bmatrix}0.9280\\0.9712\\0.9365\\0.7246\end{bmatrix},$$

$$Sup_{23}=Sup_{32}=\begin{bmatrix}0.6426\\0.8873\\0.8049\\0.8071\end{bmatrix},\quad Sup_{24}=Sup_{42}=\begin{bmatrix}0.5824\\0.9659\\0.8314\\0.8813\end{bmatrix},\quad Sup_{34}=Sup_{43}=\begin{bmatrix}0.9398\\0.9214\\0.9735\\0.9258\end{bmatrix}.$$

Step 6. Use Eq. (53) to calculate the weighted support $T'\left(r_{ij},\alpha_{ij}\right)$ of 2-tuple $\left(r_{ij},\alpha_{ij}\right)$ by the other 2-tuples, $\left(r_{ip},\alpha_{ip}\right)$ ($p=1,2,3,4$, and $p\neq j$). We denote $\left(T'\left(r_{ij},\alpha_{ij}\right)\right)_{4\times4}$ by T' in the following equation:

$$T'=\begin{bmatrix}0.5963 & 0.4735 & 0.6451 & 0.6589\\0.6661 & 0.6961 & 0.6912 & 0.7632\\0.6385 & 0.6360 & 0.6690 & 0.7322\\0.5183 & 0.6310 & 0.5821 & 0.6692\end{bmatrix}.$$

Use Eq. (54) to calculate the weights η_{ij} $(j=1,2,3,4)$ of 2-tuple $\left(r_{ij},\alpha_{ij}\right)$ $(j=1,2,3,4)$. We denote $\left(\eta_{ij}\right)_{4\times4}$ by V in the following:

$$V=\begin{bmatrix}0.3011 & 0.2316 & 0.2586 & 0.2086\\0.2941 & 0.2495 & 0.2488 & 0.2075\\0.2954 & 0.2458 & 0.2507 & 0.2082\\0.2860 & 0.2560 & 0.2483 & 0.2096\end{bmatrix}.$$

Step 7. Use the G2TLPWA operator (Eq. (55)) to aggregate all of the preference values, $\left(r_{ij},\alpha_{ij}\right)$ ($j=1,2,3,4$), in the *i*th line of $\overline{R}$ and then derive the collective overall preference value, $\overline{r}_i=\left(r_i,\alpha_i\right)$ $(i=1,2,3,4)$, of the alternative x_i $(i=1,2,3,4)$.

$$\overline{r}_1 = (s_4, 0.4882),\ \overline{r}_2 = (s_6, 0.3628),\ \overline{r}_3 = (s_6, -0.3124),\ \overline{r}_4 = (s_6, 0.0731).$$

Using Definition 2.3, we then rank the $r_i\ (i=1,2,3,4)$ in descending order:

$$\overline{r}_2 > \overline{r}_4 > \overline{r}_3 > \overline{r}_1.$$

Step 8. Rank all of the alternatives, $x_i\ (i=1,2,3,4)$, as follows:
$x_2 \succ x_4 \succ x_3 \succ x_1$.

The best alternative is x_2.

As the parameter λ varies, we may obtain different results. In Table 8, observe that the 2-tuples obtained with the G2TLPWA operator become larger as λ increases for the same aggregation arguments, and the decision makers can choose the values of λ according to their preferences.

Table 8. $\Delta^{-1}(\overline{r}_i)$ obtained with the G2TLPWA operator and rankings of the alternatives

8	$\lambda=1/3$	$\lambda=10$	$\lambda=20$	$\lambda=30$	$\lambda=40$	$\lambda=50$
x_1	3.5328	6.3167	6.9947	7.2916	7.4575	7.5617
x_2	6.0491	7.1196	7.4688	7.6292	7.7179	7.7731
x_3	5.2329	6.7403	7.2276	7.4560	7.5843	7.6648
x_4	5.4219	7.1388	7.5244	7.6767	7.7559	7.8041
Ranking	$x_2 \succ x_4 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$	$x_4 \succ x_2 \succ x_3 \succ x_1$

Furthermore, it is possible to analyze how the different attitudinal character λ plays a role in the aggregation results. To do so, we consider different values of λ, 0.01, 0.02, 0.03, 0.04, 0.05, $\cdots$, 50, which are provided by the decision maker. The result of a symbolic aggregation operation $\beta_i = \Delta^{-1}(\overline{r}_i)\quad (i=1,2,3,4)$ of the collective overall preference values $\overline{r}_i$ $(i=1,2,3,4)$ of the alternatives $x_i\ (i=1,2,3,4)$ are shown in Fig. 1.

Fig. 1 demonstrates that all $\beta_i\quad (i=1,2,3,4)$ increase as λ increases. Fig. 1 also demonstrates that as λ increases, first, x_2 is the best choice; then, x_4 is the best choice.

If the 2TLPWG operator is used in place of the G2TLPWA operator to aggregate the values of the alternatives in steps 4 and 7. The collective overall preference value, $\overline{r}_i = (r_i, \alpha_i)$ $(i=1,2,3,4)$, of the alternative $x_i\ (i=1,2,3,4)$ are as follows:

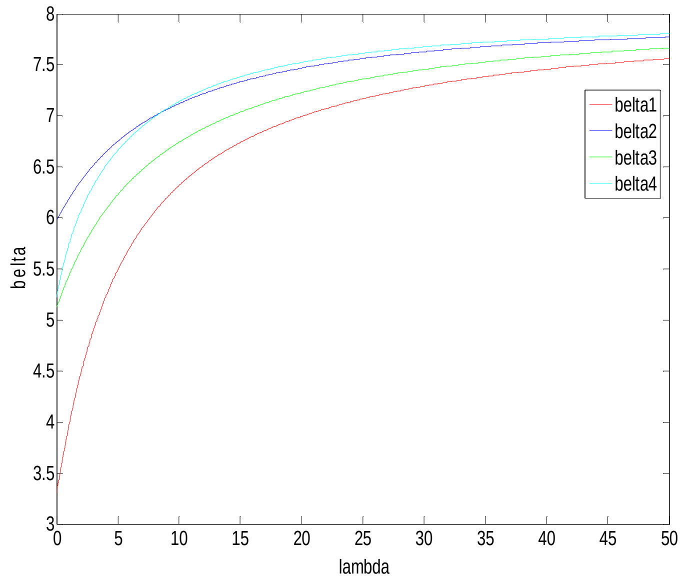

Fig. 1. Variation of β_i with respect to the parameter λ.

$\bar{r}_1 = (s_3, 0.3056)$, $\bar{r}_2 = (s_6, -0.0242)$, $\bar{r}_3 = (s_5, 0.1237)$, $\bar{r}_4 = (s_5, 0.2227)$.

Using Definition 2.3, we then rank the $\bar{r}_i$ $(i = 1,2,3,4)$ in descending order:

$\bar{r}_2 > \bar{r}_4 > \bar{r}_3 > \bar{r}_1$.

Rank all of the alternatives, x_i $(i = 1,2,3,4)$, as follows:

$x_2 \succ x_4 \succ x_3 \succ x_1$.

Thus, the best alternative is x_2.

It is clear that the 2-tuples obtained with the G2TLPWA operator are always greater than those obtained with the 2TLPWG operator for the same aggregation values and any λ.

Example 5.2 [34,43]. Let us reconsider Example 5.1. Suppose that the weights of the decision makers and the attributes are unknown; then, we use Approach II to determine the decision. Assume that $R^{(k)} = \left(r_{ij}^{(k)}\right)_{m \times n}$ $(k = 1,2,3)$ are three linguistic decision matrices shown in Tables 1-3. $\bar{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{4 \times 4}$ $(k = 1,2,3)$ are three 2-tuple linguistic decision matrices given in Tables 4-6.

Step 1. We denote $\left(r_{ij}^{(index(k))},0\right)_{4\times4}$ by $\overline{R}^{(index(k))}$, where $\left(r_{ij}^{(index(k))},0\right)$ is the *k*th largest 2-tuple of all the 2-tuples $\left(r_{ij}^{(k)},0\right)$ $(k=1,2,3)$. $\overline{R}^{(index(k))}$ $(k=1,2,3)$ are given in Tables 9-11.

Table 9. The 2-tuple linguistic decision matrix $\overline{R}^{(index(1))}$

9	c_1	c_2	c_3	c_4
x_1	$(s_5,0)$	$(s_3,0)$	$(s_7,0)$	$(s_8,0)$
x_2	$(s_7,0)$	$(s_8,0)$	$(s_8,0)$	$(s_8,0)$
x_3	$(s_7,0)$	$(s_8,0)$	$(s_7,0)$	$(s_6,0)$

Table 10. The 2-tuple linguistic decision matrix $\overline{R}^{(index(2))}$

10	c_1	c_2	c_3	c_4
x_1	$(s_5,0)$	$(s_2,0)$	$(s_2,0)$	$(s_5,0)$
x_2	$(s_7,0)$	$(s_6,0)$	$(s_6,0)$	$(s_6,0)$
x_3	$(s_5,0)$	$(s_6,0)$	$(s_5,0)$	$(s_5,0)$
x_4	$(s_8,0)$	$(s_6,0)$	$(s_5,0)$	$(s_6,0)$

Table 11. The 2-tuple linguistic decision matrix $\overline{R}^{(index(3))}$

11	c_1	c_2	c_3	c_4
x_1	$(s_4,0)$	$(s_1,0)$	$(s_1,0)$	$(s_3,0)$
x_2	$(s_3,0)$	$(s_4,0)$	$(s_5,0)$	$(s_5,0)$
x_3	$(s_3,0)$	$(s_2,0)$	$(s_3,0)$	$(s_4,0)$
x_4	$(s_6,0)$	$(s_1,0)$	$(s_3,0)$	$(s_3,0)$
x_4	$(s_8,0)$	$(s_8,0)$	$(s_5,0)$	$(s_7,0)$

Step 2. We use Eq. (57) to calculate the supports $Sup\left(\left(r_{ij}^{(index(k))},0\right),\left(r_{ij}^{(index(t))},0\right)\right)$ ($i,j=1,2,3,4$, $k,t=1,2,3$, $k\neq t$). For simplicity, we denote $\left(Sup\left(\left(r_{ij}^{(index(k))},0\right),\left(r_{ij}^{(index(t))},0\right)\right)\right)_{4\times4}$ by $Supp^{kt}$, which refers to the supports between $\overline{R}^{(index(k))}$ and $\overline{R}^{(index(t))}$, in the following:

$$Supp^{12} = Supp^{21} = \begin{bmatrix} 1.0000 & 0.8750 & 0.3750 & 0.6250 \\ 1.0000 & 0.7500 & 0.7500 & 0.7500 \\ 0.7500 & 0.7500 & 0.7500 & 0.8750 \\ 1.0000 & 0.7500 & 1.0000 & 0.8750 \end{bmatrix},$$

$$Supp^{13} = Supp^{31} = \begin{bmatrix} 0.8750 & 0.7500 & 0.2500 & 0.3750 \\ 0.5000 & 0.5000 & 0.6250 & 0.6250 \\ 0.5000 & 0.2500 & 0.5000 & 0.7500 \\ 0.7500 & 0.1250 & 0.7500 & 0.5000 \end{bmatrix},$$

$$Supp^{23} = Supp^{32} = \begin{bmatrix} 0.8750 & 0.8750 & 0.8750 & 0.7500 \\ 0.5000 & 0.7500 & 0.8750 & 0.8750 \\ 0.7500 & 0.5000 & 0.7500 & 0.8750 \\ 0.7500 & 0.3750 & 0.7500 & 0.6250 \end{bmatrix}.$$

Step 3. Use Eq. (59) to calculate the weighted support $T\left(r_{ij}^{(index(k))},0\right)$ of the *k*th largest 2-tuple $\left(r_{ij}^{(index(k))},0\right)$ by the other 2-tuples $\left(r_{ij}^{(index(t))},0\right)$ ($t=1,2,3$, $t \neq k$). We denote $\left(T\left(r_{ij}^{(index(k))},0\right)\right)_{4\times4}$ by T_k ($k=1,2,3$) in the following:

$$T_1 = \begin{bmatrix} 1.8750 & 1.6250 & 0.6250 & 1.0000 \\ 1.5000 & 1.2500 & 1.3750 & 1.3750 \\ 1.2500 & 1.0000 & 1.2500 & 1.6250 \\ 1.7500 & 0.8750 & 1.7500 & 1.3750 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1.8750 & 1.7500 & 1.2500 & 1.3750 \\ 1.5000 & 1.5000 & 1.6250 & 1.6250 \\ 1.5000 & 1.2500 & 1.5000 & 1.7500 \\ 1.7500 & 1.1250 & 1.7500 & 1.5000 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 1.7500 & 1.6250 & 1.1250 & 1.1250 \\ 1.0000 & 1.2500 & 1.5000 & 1.5000 \\ 1.2500 & 0.7500 & 1.2500 & 1.6250 \\ 1.5000 & 0.5000 & 1.5000 & 1.1250 \end{bmatrix}.$$

Let $g(x)=x^3$ and use Eq. (60) to calculate the weights $u_{ij}^{(k)}$ $(i,j=1,2,3,4,\ k=1,2,3)$ associated with the *k*th largest 2-tuple $\left(r_{ij}^{(index(k))},0\right)$ $(i,j=1,2,3,4,\ k=1,2,3)$. We denote $\left(u_{ij}^{(k)}\right)_{4\times4}$ by V_k ($k=1,2,3$) in the following:

$$V_1 = \begin{bmatrix} 0.0387 & 0.0353 & 0.0199 & 0.0291 \\ 0.0456 & 0.0332 & 0.0318 & 0.0318 \\ 0.0332 & 0.0370 & 0.0332 & 0.0353 \\ 0.0406 & 0.0396 & 0.0406 & 0.0391 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.2709 & 0.2680 & 0.2495 & 0.2758 \\ 0.3189 & 0.2792 & 0.2645 & 0.2645 \\ 0.2792 & 0.3184 & 0.2792 & 0.2680 \\ 0.2843 & 0.3451 & 0.2843 & 0.2987 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} 0.6904 & 0.6967 & 0.7306 & 0.6951 \\ 0.6356 & 0.6875 & 0.7037 & 0.7037 \\ 0.6875 & 0.6446 & 0.6875 & 0.6967 \\ 0.6750 & 0.6153 & 0.6750 & 0.6622 \end{bmatrix}.$$

Step 4. Let $\lambda = 0.7$. Use the G2TLPOWA operator (Eq. (61)) to aggregate all the individual 2-tuple linguistic decision matrices $\overline{R}^{(k)} = \left(\left(r_{ij}^{(k)}, 0\right)\right)_{4\times4}$ ($k = 1,2,3$) into the collective 2-tuple linguistic decision matrix $\overline{R} = \left(\overline{r}_{ij}\right)_{4\times4} = \left(\left(r_{ij}, \alpha_{ij}\right)\right)_{4\times4}$ (see Table 12).

Table 12. The collective 2-tuple linguistic decision matrix $\overline{R}$

12	c_1	c_2	c_3	c_4
x_1	$(s_4, 0.3024)$	$(s_1, 0.3103)$	$(s_1, 0.3185)$	$(s_4, -0.3493)$
x_2	$(s_4, 0.3404)$	$(s_5, -0.3426)$	$(s_5, 0.3491)$	$(s_5, 0.3491)$
x_3	$(s_4, -0.3510)$	$(s_3, 0.3313)$	$(s_4, -0.3510)$	$(s_4, 0.3290)$
x_4	$(s_7, -0.3691)$	$(s_3, -0.3185)$	$(s_4, -0.3839)$	$(s_4, -0.0229)$

Step 5. We denote $\left(r_{iindex(j)}, \alpha_{iindex(j)}\right)_{4\times4}$ by $\overline{R}_{index}$, where $\left(r_{iindex(j)}, \alpha_{iindex(j)}\right)$ is the *j*th largest 2-tuple of all the 2-tuples $\left(r_{iq}, \alpha_{iq}\right)(q = 1,2,3,4)$. $\overline{R}_{index}$ is shown in Table 13.

Table 13. The 2-tuple linguistic decision matrix $\overline{R}_{index}$

13	c_1	c_2	c_3	c_4
x_1	$(s_4, 0.3024)$	$(s_4, -0.3493)$	$(s_1, 0.3185)$	$(s_1, 0.3103)$
x_2	$(s_5, 0.3491)$	$(s_5, 0.3491)$	$(s_5, -0.3426)$	$(s_4, 0.3404)$
x_3	$(s_4, 0.3290)$	$(s_4, -0.3510)$	$(s_4, -0.3510)$	$(s_3, 0.3313)$
x_4	$(s_7, -0.3691)$	$(s_4, -0.0229)$	$(s_4, -0.3839)$	$(s_3, -0.3185)$

Use Eq. (63) to calculate the supports $Sup\left(\left(r_{iindex(j)},\alpha_{iindex(j)}\right),\left(r_{iindex(p)},\alpha_{iindex(p)}\right)\right)$ ($i=1,2,3,4$, $j,p=1,2,3,4$, $j\neq p$). For simplicity, we denote $\left(Sup\left(\left(r_{iindex(j)},\alpha_{iindex(j)}\right),\left(r_{iindex(p)},\alpha_{iindex(p)}\right)\right)\right)_{4\times1}$ by $Supp_{jp}$, which refers to the supports between the *j*th row and the *p*th row of $\bar{R}_{index}$, in the following:

$$Supp_{12}=Supp_{21}=\begin{bmatrix}0.9185\\1.0000\\0.9150\\0.6683\end{bmatrix},\quad Supp_{13}=Supp_{31}=\begin{bmatrix}0.6270\\0.9135\\0.9150\\0.6232\end{bmatrix},$$

$$Supp_{14}=Supp_{41}=\begin{bmatrix}0.6260\\0.8739\\0.8753\\0.5063\end{bmatrix},\quad Supp_{23}=Supp_{32}=\begin{bmatrix}0.7085\\0.9135\\1.0000\\0.9549\end{bmatrix},$$

$$Supp_{24}=Supp_{42}=\begin{bmatrix}0.7074\\0.8739\\0.9603\\0.8381\end{bmatrix},\quad Supp_{34}=Supp_{43}=\begin{bmatrix}0.9990\\0.9604\\0.9603\\0.8832\end{bmatrix}.$$

Step 6. Use Eq. (65) to calculate the weighted support $T\left(r_{iindex(j)},\alpha_{iindex(j)}\right)$ of the *j*th largest 2-tuple $\left(r_{iindex(j)},\alpha_{iindex(j)}\right)$ by the other 2-tuples $\left(r_{iindex(p)},\alpha_{iindex(p)}\right)$ ($p=1,2,3,4$, and $p\neq j$). We denote $\left(T\left(r_{iindex(j)},\alpha_{iindex(j)}\right)\right)_{4\times4}$ by T in the following:

$$T=\begin{bmatrix}2.1715 & 2.3345 & 2.3345 & 2.3324\\2.7874 & 2.7874 & 2.7874 & 2.7082\\2.7053 & 2.8753 & 2.8753 & 2.7959\\1.7978 & 2.4612 & 2.4612 & 2.2276\end{bmatrix}.$$

Use Eq. (66) to calculate the weights u_{ij} $(j=1,2,3,4)$ associated with the *j*th largest 2-tuple $\left(r_{iindex(j)},\alpha_{iindex(j)}\right)$ $(j=1,2,3,4)$. We denote $\left(u_{ij}\right)_{4\times4}$ by V in the following:

$$V=\begin{bmatrix}0.0140 & 0.1065 & 0.2964 & 0.5831\\ 0.0159 & 0.1111 & 0.3016 & 0.5714\\ 0.0143 & 0.1084 & 0.3010 & 0.5762\\ 0.0101 & 0.1029 & 0.3101 & 0.5769\end{bmatrix}.$$

Step 7. Use the G2TLPOWA operator (Eq. (67)) to aggregate all the preference values (r_{ij},α_{ij}) ($j=1,2,3,4$) in the *i*th line of $\overline{R}$. Then, derive the collective overall preference value $\overline{r}_i=(r_i,\alpha_i)$ $(i=1,2,3,4)$ of the alternative x_i $(i=1,2,3,4)$ as follows:

$$\overline{r}_1=(s_2,-0.4375),\ \overline{r}_2=(s_5,-0.4394),\ \overline{r}_3=(s_3,0.4742),\ \overline{r}_4=(s_3,0.1279).$$

Step 8. According to Definition 2.3, we rank $\overline{r}_i$ $(i=1,2,3,4)$ in descending order:

$$\overline{r}_2>\overline{r}_3>\overline{r}_4>\overline{r}_1.$$

Step 9. Rank all the alternatives x_i $(i=1,2,3,4)$ as follows:

$$x_2\succ x_3\succ x_4\succ x_1.$$

Thus, the best alternative is x_2.

As the parameter λ changes, we obtain different results (see Table 14). From Table 14, we see that the 2-tuples obtained by the G2TLPOWA operator increase as the parameter λ increases and the aggregation arguments are kept fixed. The decision makers can choose the value of λ according to their preferences.

Table 14. $\Delta^{-1}(\overline{r}_i)$ obtained by the G2TLPOWA operator and the rankings of alternatives

14	$\lambda=1/3$	$\lambda=10$	$\lambda=20$	$\lambda=30$	$\lambda=40$	$\lambda=50$
x_1	1.4774	4.3556	5.5256	6.1998	6.5958	6.8524
x_2	4.4632	5.9488	6.6950	6.9902	7.1995	7.3461
x_3	3.3418	4.9573	5.7959	6.3031	6.6433	6.8826
x_4	2.8579	5.4536	6.3754	6.8595	7.1283	7.2958
Ranking	$x_2\succ x_3\succ x_4\succ x_1$	$x_2\succ x_4\succ x_3\succ x_1$	$x_2\succ x_4\succ x_3\succ x_1$	$x_2\succ x_4\succ x_3\succ x_1$	$x_2\succ x_4\succ x_3\succ x_1$	$x_2\succ x_4\succ x_3\succ x_1$

Furthermore, it is possible to analyze how different values of the attitudinal character λ change the aggregation results. To do so, we consider different value of λ, 0.01, 0.02, 0.03, 0.04, 0.05, $\cdots$, 50. The result of a symbolic aggregation operation $\beta_i = \Delta^{-1}(\overline{r}_i)$ $(i=1,2,3,4)$ of the collective overall preference values $\overline{r}_i$ $(i=1,2,3,4)$ of the alternatives x_i $(i=1,2,3,4)$ are shown in Fig. 2.

Fig. 2 demonstrates that all of the β_i $(i=1,2,3,4)$ increase as λ increases. From Fig. 2, we can see that as λ increases, x_2 is always the best choice.

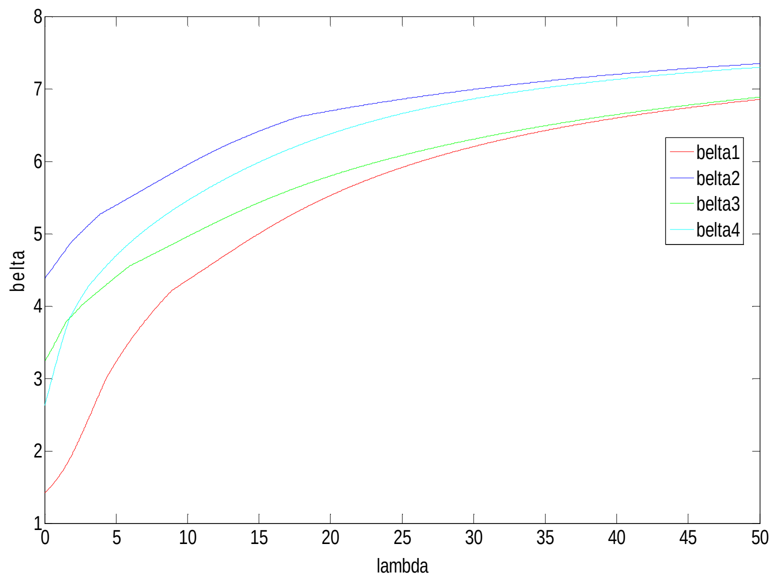

Fig. 2. Variation of β_i with respect to the parameter λ.

In the above example, if we use the 2TLPOWG operator instead of the G2TLPOWA operator to aggregate the values of the alternatives in steps 4 and 7, then the collective overall preference value, $\overline{r}_i=(r_i,\alpha_i)$ $(i=1,2,3,4)$, of the alternative x_i $(i=1,2,3,4)$ are as follows:

$$\overline{r}_1=(s_1,0.4116),\ \overline{r}_2=(s_4,0.3773),\ \overline{r}_3=(s_3,0.2287),\ \overline{r}_4=(s_3,-0.3847).$$

Using Definition 2.3, we then rank the $\overline{r}_i$ $(i=1,2,3,4)$ in descending order:

$$\overline{r}_2 > \overline{r}_3 > \overline{r}_4 > \overline{r}_1.$$

Rank all of the alternatives, x_i $(i=1,2,3,4)$, as follows:

$$x_2 \succ x_3 \succ x_4 \succ x_1.$$

Thus, the best alternative is x_2.

It is clear that the 2-tuples obtained with the G2TLPOWA operator are always greater than those obtained with the 2TLPOWG operator for the same aggregation values and any λ.

6. CONCLUSIONS

In this paper, we have developed several new 2-tuple linguistic power aggregation operators, including the 2TLPG, 2TLPWG, 2TLPOWG, G2TLPA, G2TLPWA, and G2TLPOWA operators. We have studied some fundamental properties of the developed operators, such as commutativity, idempotency, boundedness, and monotonicity. The primary advantage of these operators is that they take the relationships between the 2-tuples being aggregated into account. Furthermore, we have used the proposed operators to develop two approaches to multiple attribute group decision making with 2-tuple linguistic information. Concretely, if the weight vectors of the decision makers and attributes are known, then we employ an approach based on the G2TLPWA and 2TLPWG operators to aggregate all of the individual 2-tuple linguistic decision matrices into a collective 2-tuple linguistic decision matrix and then utilize the G2TLPWA and 2TLPWG operators to derive the collective overall preference values of each alternative. If the weight vectors of the decision makers and attributes are unknown, then we employ another approach based on the G2TLPOWA and 2TLPOWG operators to aggregate the individual 2-tuple linguistic decision matrices and utilize the G2TLPOWA and 2TLPOWG operators to derive the collective overall preference values of each alternative. Our approaches incorporate all of the decision arguments as well as the relationships between them. Finally, two numerical examples are provided to illustrate the developed approaches.

ACKNOWLEDGEMENTS

The author thanks the anonymous referees for their valuable suggestions in improving this paper. This work is supported by the National Natural Science Foundation of China (Grant Nos. 61073121 and 61375075) and the Natural Science Foundation of Hebei Province of China (Grant Nos. F2012201020 and A2012201033).

COMPETING INTERESTS

Author has declared that no competing interests exist.

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