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The Approximation of Hermite Interpolation on the Weighted Mean Norm

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Abstract

We research the simultaneous approximation problem of the higher-order Hermite interpolation based on the zeros of the second Chebyshev polynomials under weighted Lp-norm. The estimation is sharp.

Keywords

Hermite Interpolation Operator, Chebyshev Polynomial, Derivative Approximation

1. Introduction

For $0 < p < +\infty$ and a non-negative measurable function *u*, the space L_u^p is defined to be the set of measurable f, such that

$$\|f\|_{p,u} = \left(\int_{-1}^{1} |f(t)|^p u(t)\,\mathrm{d}t\right)^{1/p}, \quad 0 < p < +\infty$$

is finite. Of course, when $0 < p < 1$, $\|\bullet\|_{p,u}$ is not a norm; nevertheless, we keep this notation for convenience. For $u = 1$, this is the usual L^p space. For $d \in N$, we write C^d for the space of functions that have *d*th continuous derivative on $[-1,1]$.

We introduce a few notations. If ω is a Jacobi weight function, we write $\omega \in J$. Let $\omega \in J, \omega(x) = \omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$. The Jacobi polynomials $p_n(\omega)$ are orthogonal polynomials with respect to the weight function ω, *i.e.*

$$\int_{-1}^{1} p_n(\omega, x) p_m(\omega, x)\omega(x)\,\mathrm{d}x = \delta_{n,m}$$

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It is well known that $p_n(\omega)$ has n distinct zeros in $(-1,1)$. These zeros are denoted by $x_{kn}(\omega)$ and the following order is assumed:

$$1 > x_{1n}(\omega) > x_{2n}(\omega) > \cdots > x_{nn}(\omega) > -1$$

Later, when we fix ω, we shall write x_{kn} instead of $x_{kn}(\omega)$.

For a given integer $r \ge 0, s \ge 0$ and $m \ge 1$, the Hermite interpolation is defined to be the unique polynomial of degree $N = mn + r + s - 1$, denoted by $H_{n,m,r,s}(\omega, f)$, satisfying

$$\begin{cases} H_{n,m,r,s}^{(t)}(\omega, f, x_{kn}) = f^{(t)}(x_{kn}), 0 \le t \le m-1, 1 \le k \le n; \\ H_{n,m,r,s}^{(t)}(\omega, f, 1) = f^{(t)}(1), 0 \le t \le r-1; \\ H_{n,m,r,s}^{(t)}(\omega, f, -1) = f^{(t)}(-1), 0 \le t \le s-1 \end{cases}$$

for $f \in C^M$, where $M = \max\{m-1, r-1, s-1\}$, if $r = 0$ or $s = 0$ then we have no interpolation at 1 or −1. We shall fix the integers m, r and s for the rest of the paper, and omit them from the notations. Thus, for example, we shall write $H_n(\omega, f)$ instead of $H_{n,m,r,s}(\omega, f)$. Let

$$\varphi(x) = \sqrt{1-x^2}, \ \omega_m^{(r,s)} := \left[(1-x)^{\alpha - \frac{2r}{m} + \frac{1}{2}} (1+x)^{\beta - \frac{2s}{m} + \frac{1}{2}}\right]^{\frac{m}{2}}.$$

Vertesi and Xu [1], Nevai and Xu [2], and Pottinger considered the simultaneous approximation by Hermite interpolation operators.

We have researched the simultaneous approximation problem of the lower-order Hermite interpolation based on the zeros of Chebyshev polynomials under weighted Lp-norm in references [3]-[5]. We will research the simultaneous approximation problem of the higher-order Hermite interpolation in this article.

Let

$$X_n = \left\{x_k = \cos\theta_k = \cos\frac{k\pi}{n+1} : 1 \le k \le n\right\}$$

be the zeros of $U_n(x) = \dfrac{\sin(n+1)\theta}{\sin\theta}, x = \cos\theta$, the nth degree Chebyshev polynomial of the second kind. For $f \in C_{[-1,1]}^2$, let $H_n(f, x)$ be the polynomial of degree at most $3n-1$ which satisfies

$$H_n^{(t)}(f, x_k) = f^{(t)}(x_k), t = 0,1,2, k = 1,2,\cdots,n$$

Then the Hermite interpolation polynomial is given by

$$H_n(f,x) = \sum_{k=1}^{n} f(x_k) L_{k,0}(x) + \sum_{k=1}^{n} f'(x_k) L_{k,1}(x) + \sum_{k=1}^{n} f''(x_k) L_{k,2}(x) \tag{1.1}$$

where

$$L_{k,0}(x) = l_k^3(x) - \frac{9x_k}{2(1-x_k^2)}(x-x_k) l_k^3(x) + \frac{1}{2}\left(\frac{12x_k^2}{(1-x_k^2)^2} + \frac{n^2+2n-3}{1-x_k^2}\right)(x-x_k)^2 l_k^3(x) \tag{1.2}$$

$$L_{k,1}(x) = (x-x_k) l_k^3(x) - \frac{9x_k}{2(1-x_k^2)}(x-x_k)^2 l_k^3(x) \tag{1.3}$$

$$L_{k,2}(x) = \frac{(x-x_k)^2}{2} l_k^3(x) \tag{1.4}$$

$$l_k(x) = \frac{U_n(x)}{U_n'(x)(x-x_k)} = \frac{(-1)^k (1-x_k^2) U_n(x)}{(n+1)(x-x_k)} \tag{1.5}$$

Theorem 1.
Let $H_n(f,x)$ be defined as (1.1), for $f \in C^2_{[-1,1]}$ and $p>0, \alpha>-1$, then we have

$$\int_{-1}^{1}\left|H_n'(f,x)-f'(x)\right|^p\left(1-x^2\right)^\alpha \mathrm{d}x \le \begin{cases} Cn^{p-2\alpha-2}E_{3n-3}^{p}(f''), & p-2\alpha-1>1; \\ C\ln nE_{3n-3}^{p}(f''), & p-2\alpha-1=1. \end{cases}$$

2. Some Lemmas

Lemmas 1. [6] Let $H_n(f,x)$ be defined as (1.1), then

$$L_{k,h}(x)=\frac{A(x)}{(x-x_k)^{\alpha_k}}\cdot\frac{(x-x_k)^h}{h!}\cdot\left\{\frac{(x-x_k)^{\alpha_k}}{A(x)}\right\}_{(x-x_k)}^{(\alpha_k-h-1)}$$

where $A(x)=\prod_{k=1}^{n}(x-x_k)^{\alpha_k}, \alpha_k \in N$, $\alpha_1+\alpha_2+\cdots+\alpha_n=m+1$, $\left\{\frac{(x-x_k)^{\alpha_k}}{A(x)}\right\}_{(x-x_k)}^{(\alpha_k-h-1)}$ is defined as function $\frac{(x-x_k)^{\alpha_k}}{A(x)}$ at $x=x_k$ before the commencement of the Taylor series of α_k-h.

Lemma 2. [7]
If $f\in C^2_{[-1,1]}$, then there exists a algebraic polynomial $p_{3n-1}(x)$ of degree at most $3n-1$ such that

$$\left|f^{(i)}(x)-p_{3n-1}^{(i)}(x)\right|\le C\left[\frac{\sqrt{1-x^2}}{n}\right]^{2-i}E_{3n-2}(f''). \quad i=0,1,2$$

Let

$$-1=t_{2n}<t_{2n-1}<\cdots<t_1<t_0=1$$

be the zeros of $(1-x^2)U_{2n-1}(x)$, here $U_{n-1}(x)=\frac{\sin n\theta}{\sin\theta}, x=\cos\theta$, the nth degree Chebyshev polynomial of the second kind. For $f\in C^1_{[-1,1]}$, the well-known Lagrange interpolation polynomial of f based on $\{t_k\}_{k=0}^{2n}$ is given by

$$Q_{2n}(f,x)=\sum_{k=0}^{2n}f(t_k)\varphi_k(x) \tag{2.1}$$

where

$$\varphi_0(x)=\frac{(1+x)U_{2n-1}(x)}{2U_{2n-1}(1)} \tag{2.2}$$

$$\varphi_{2n}(x)=\frac{(1-x)U_{2n-1}(x)}{2U_{2n-1}(-1)} \tag{2.3}$$

$$\varphi_k(x)=\frac{(-1)^{k+1}(1-x^2)U_{2n-1}(x)}{2n(x-t_k)}, k=1,\cdots,2n-1 \tag{2.4}$$

Lemma 3. [7] Let $\varphi_k(x), k=0,1,\cdots,2n$ be defined as (2.4), for $\alpha,\beta>-1$, and $p>0$, we have

$$\left(\int_{-1}^{1}\left|\sum_{k=1}^{2n-1}A_k\varphi_k(x)\right|^p(1-x)^\alpha(1+x)^\beta \mathrm{d}x\right)^{\frac{1}{p}}\le C\max_{1\le k\le 2n-1}|A_k|.$$

3. The Proof of Theorem 1

For $f \in C^2_{[-1,1]}$, let $p_{3n-1}(x)$ be the polynomial of degree at most $3n-1$ which satisfies Lemma 2. By the uniqueness of Hemite interpolation polynomial, it can be easily checked that,

$$H_n'(f,x) - f'(x) = H_n'(f - p_{3n-1}, x) + p_{3n-1}'(x) - f'(x) \tag{3.1}$$

We can conclude that

$$\begin{aligned} I &= \int_{-1}^{1} \left| H_n'(f,x) - f'(x) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &\le 2^p \int_{-1}^{1} \left| H_n'(f - p_{3n-1}, x) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x + 2^p \int_{-1}^{1} \left| p_{3n-1}'(x_k) - f'(x_k) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &= 2^p \left(I_1 + I_2\right). \end{aligned} \tag{3.2}$$

Firstly, we estimate I_1. By (3.1), we have

$$\begin{aligned} I_1 &\le 3^p \int_{-1}^{1} \left| \sum_{k=1}^{n} \left(f(x_k) - P_{3n-1}(x_k) \right) L_{k,0}'(x) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &\quad + 3^p \int_{-1}^{1} \left| \sum_{k=1}^{n} \left(f'(x_k) - P_{3n-1}'(x_k) \right) L_{k,1}'(x) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &\quad + 3^p \int_{-1}^{1} \left| \sum_{k=1}^{n} \left(f''(x_k) - P_{3n-1}''(x_k) \right) L_{k,2}'(x) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &= 3^p \left(I_{11} + I_{12} + I_{13}\right). \end{aligned} \tag{3.3}$$

Firstly, we estimate I_{11},

$$\begin{aligned} I_{11} &\le 3^p \int_{-1}^{1} \left| \sum_{k=1}^{n} \left(f(x_k) - P_{3n-1}(x_k) \right) \left(3l_k^2(x) l_k'(x) - \frac{9x_k}{2\left(1-x_k^2\right)} l_k^3(x) \right) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &\quad + 3^p \int_{-1}^{1} \left| \sum_{k=1}^{n} \left(f(x_k) - P_{3n-1}(x_k) \right) \left(-\frac{27x_k}{2\left(1-x_k^2\right)} (x-x_k) l_k^2(x) l_k'(x) \right. \right. \\ &\quad \left. \left. + \frac{3}{2} \left(\frac{12x_k^2}{\left(1-x_k^2\right)^2} + \frac{n^2+2n-3}{1-x_k^2} \right) (x-x_k)^2 l_k^2(x) l_k'(x) \right) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &\quad + 3^p \int_{-1}^{1} \left| \sum_{k=1}^{n} \left(f(x_k) - P_{3n-1}(x_k) \right) \left(\frac{12x_k^2}{\left(1-x_k^2\right)^2} + \frac{n^2+2n-3}{1-x_k^2} \right) (x-x_k) l_k^3(x) \right|^p \left(1-x^2\right)^\alpha \mathrm{d}x \\ &= 3^p \left(I_A + I_B + I_C\right). \end{aligned} \tag{3.4}$$

Let

$$B(x) = \sum_{k=1}^{n} \left(f(x_k) - P_{3n-1}(x_k) \right) (-1)^k \left(1-x_k^2\right) U_n(x) \left(\frac{l_k(x) l_k'(x)}{(x-x_k)} - \frac{3x_k l_k^2(x)}{2\left(1-x_k^2\right)(x-x_k)} \right) \tag{3.5}$$

be the polynomial of degree $2n-3$. By the uniqueness of Lagrange interpolation polynomial, it can be easily checked that,

$$B(x) = \sum_{l=0}^{2n} B(t_l) \varphi_l(x) \tag{3.6}$$

By (3.5), (3.6) and Lemma 3, we can derive

$$I_A \le \frac{C}{(n+1)^p}\left(\max_{0\le l\le 2n-1}\left|\sum_{k=1}^{n}\left(f(x_k)-P_{3n-1}(x_k)\right)(-1)^k\left(1-x_k^2\right)U_n(t_l)\left(\frac{l_k(t_l)l_k'(t_l)}{(t_l-x_k)}-\frac{3x_k l_k^2(t_l)}{2\left(1-x_k^2\right)(t_l-x_k)}\right)\right|\right)^p$$

$$+\frac{3^{2p}}{(n+1)^p}\int_{-1}^{1}\left|\sum_{k=1}^{n}\left(f(x_k)-P_{3n-1}(x_k)\right)(-1)^k\left(1-x_k^2\right)U_n(x)\left(\frac{l_k(1)l_k'(1)}{(1-x_k)}-\frac{3x_k l_k^2(1)}{2\left(1-x_k^2\right)(1-x_k)}\right)\right|^p$$

$$\cdot\left|\frac{(1+x)U_{2n-1}(x)}{2U_{2n-1}(1)}\right|^p\left(1-x^2\right)^\alpha \mathrm{d}x \tag{3.7}$$

$$+\frac{3^{2p}}{(n+1)^p}\int_{-1}^{1}\left|\sum_{k=1}^{n}\left(f(x_k)-P_{3n-1}(x_k)\right)(-1)^k\left(1-x_k^2\right)U_n(x)\left(\frac{l_k(-1)l_k'(-1)}{(-1-x_k)}+\frac{3x_k l_k^2(-1)}{2\left(1-x_k^2\right)(1+x_k)}\right)\right|^p$$

$$\cdot\left|\frac{(1+x)U_{2n-1}(x)}{2U_{2n-1}(-1)}\right|^p\left(1-x^2\right)^\alpha \mathrm{d}x$$

$$= M_1 + 3^{2p}\left(M_2+M_3\right).$$

Firstly, we estimate M_1. Let

$$A(l)=\left|\sum_{k=1}^{n}\left(f(x_k)-P_{3n-1}(x_k)\right)(-1)^k\left(1-x_k^2\right)U_n(x)\left(\frac{l_k(x)l_k'(x)}{(x-x_k)}-\frac{3x_k l_k^2(x)}{2\left(1-x_k^2\right)(x-x_k)}\right)\right| \tag{3.8}$$

then

$$l_k'(t_l)=\begin{cases}\dfrac{3x_k}{2\left(1-x_k^2\right)}, & l=2s, s=k;\\[2ex] \dfrac{(-1)^{s+k}\left(1-x_k^2\right)}{\left(1-t_l^2\right)(t_l-x_k)}, & l=2s, s\ne k;\\[2ex] \dfrac{(-1)^{s+k+1}\left(1-x_k^2\right)}{n+1}\cdot\dfrac{t_l}{(t_l-x_k)^2\left(1-t_l^2\right)^{\frac{3}{2}}}, & l=2s-1.\end{cases} \tag{3.9}$$

From Lemma 2 and (3.8), (3.9), we have that for $l=2s-1$.

$$A(l)\le CE_{3n-3}(f'') \tag{3.10}$$

For $l=2s, s=1,2,\cdots,n$, we have

$$A(l)=0 \tag{3.11}$$

We can conclude

$$M_1\le CE_{3n-3}^p(f''). \tag{3.12}$$

Secondly, we estimate M_2, and by Lemma 2, we get

$$M_2\le\begin{cases}Cn^{p-2\alpha-2}E_{3n-3}^p(f'') & p-2\alpha-1>1\\ C\ln nE_{3n-3}^p(f'') & p-2\alpha-1=1\end{cases} \tag{3.13}$$

Similarly

$$M_3\le\begin{cases}Cn^{p-2\alpha-2}E_{3n-3}^p(f'') & p-2\alpha-1>1\\ C\ln nE_{3n-3}^p(f'') & p-2\alpha-1=1\end{cases} \tag{3.14}$$

By (3.12), (3.13) and (3.14), we have

$$I_A \le \begin{cases} Cn^{p-2\alpha-2} E_{3n-3}^p(f'') & p-2\alpha-1>1 \\ C\ln n E_{3n-3}^p(f'') & p-2\alpha-1=1 \end{cases} \tag{3.15}$$

Similarly, we get

$$I_B \le \begin{cases} Cn^{p-2\alpha-2} E_{3n-3}^p(f'') & p-2\alpha-1>1 \\ C\ln n E_{3n-3}^p(f'') & p-2\alpha-1=1 \end{cases} \tag{3.16}$$

$$I_C \le \begin{cases} Cn^{p-2\alpha-2} E_{3n-3}^p(f'') & p-2\alpha-1>1 \\ C\ln n E_{3n-3}^p(f'') & p-2\alpha-1=1 \end{cases} \tag{3.17}$$

By (3.15), (3.16) and (3.17), we get

$$I_{11} \le \begin{cases} Cn^{p-2\alpha-2} E_{3n-3}^p(f'') & p-2\alpha-1>1 \\ C\ln n E_{3n-3}^p(f'') & p-2\alpha-1=1 \end{cases} \tag{3.18}$$

Similarly, we get

$$I_{12} \le CE_{3n-3}^p(f''). \tag{3.19}$$

$$I_{13} \le CE_{3n-3}^p(f''). \tag{3.20}$$

Secondly, we estimate I_2, from Lemma 2,

$$I_2 \le CE_{3n-3}^p(f''). \tag{3.21}$$

From (3.2), (3.3), and (3.21), we can obtain the upper estimate

$$I \le \begin{cases} Cn^{p-2\alpha-2} E_{3n-3}^p(f'') & p-2\alpha-1>1 \\ C\ln n E_{3n-3}^p(f'') & p-2\alpha-1=1 \end{cases}$$

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