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Coupling of Laplace Transform and Differential Transform for Wave Equations

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, we apply Differential Transform Method (DTM) coupled with Laplace Transform Method to solve wave equations and wave-like equations which arise very frequently in physical problems related to engineering and applied sciences. It is observed that the proposed technique works very well and gives rapidly converging series solutions. Several examples are given to re-confirm the efficiency of the suggested algorithm. The graphs were performed by using Mathematica-8.

Keywords: DTM; Laplace transformation; wave equations; Mathematica.

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1 INTRODUCTION

In recent years, the rapid development of nonlinear sciences [1-10] witnesses a wide range of analytical and numerical techniques by various engineering and scientific applications. Most of the developed schemes have focused on their limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems [1-10].

Different type of schemes and analytical tool have been proposed for solving wave-like equations with variable co-efficents [1], linear and non-linear wave equations [2] and wave systems [3] etc.

The Differential Transform Method (DTM) is one of them techniques. It is an iterative procedure for obtaining analytic Taylor's series solution of differential equations which was first proposed by J.K. Zhou [4] in 1986, its main application concern with linear and non-linear initial value problems in electric circuit analysis. The Laplace-differential transform method (LDTM) is an approximate analytical technique for solving differential equations that recently introduced by Marwan Alquran *et* al. [6] and it has been successfully applied to solve different types of physical problems by Kiranta *et* al. [7-8].

The main goal of this work is the coupling of differential transform method and Laplace transformation to obtain exact solutions of wave-like equations. The suggested algorithm is tested on linear, nonlinear wave equations and wave-like equations in bounded and unbounded domains. To the best of our knowledge no such try has been made to combine LTM and DTM for solving wave equations. Four problems for wave equations and wave-like equation are solved to make clear the application of the transform and the numerical results are very encouraging.

2 DIFFERENTIAL TRANSFORMATION METHOD

The one variable differential transform [7-8] of a function $u(x,t)$, is defined as:

$$U_k(t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0}; k \geq 0 \tag{2.1}$$

where $u(x,t)$ is the original function and $U_k(t)$ is the transformed function. The inverse differential transform of $U_k(t)$ is defined as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x-x_0)^k, \tag{2.2}$$

where x_0 is the initial point for the given initial condition. Then the function $u(x,t)$ can be written as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k. \tag{2.3}$$

Some basic formulas of Differential Transformation are listed in Table 1.

3 DIFFERENTIAL TRANSFORM METHOD COUPLED WITH LAPLACE TRANSFORM

Consider the general nonlinear, inhomogeneous partial differential equation with a variable coefficient given by

$$\mathcal{L}[u] = N[u] + R[u] + \phi; \quad x \in \mathrm{I\!R}, \ \mathrm{t} \in \mathrm{I\!R}^+ \tag{3.1}$$

subject to the initial conditions

$$u(x,0) = g_0(x), \ \ u_t(x,0) = g_1(x), \ \ u_t^{(2)}(x,0) = g_2(x)..., u_t^{(n-1)}(x,0) = g_{n-1}(x) \tag{3.2}$$

and spatial conditions

$$u(0,t) = h_0(t), \ \ u_x(0,t) = h_1(t), \tag{3.3}$$

where $\mathcal{L}$ is the n^{th} order derivative w.r.t 't', $R[.]$ is linear operator and $N[.]$ is non-linear operator with second degree. And $\phi = \phi(x,t)$ and $u = u(x,t)$ are known and unknown functions respectively.

Table 1

Original Function	Transformed Function
$u(x,t) = f(x,t) \pm g(x,t)$	$U_k(t) = F_k(t) \pm G_k(t)$
$u(x,t) = \alpha f(x,t)$	$U_k(t) = \alpha F_k(t)$
$u(x,t) = \frac{\partial f(x,t)}{\partial x}$	$U_k(t) = (k+1)F_{k+1}(t)$
$u(x,t) = \frac{\partial^r f(x,t)}{\partial x^r}$	$U_k(t) = (k+1)(k+2)...(k+r)F_{k+r}(t)$
$u(x,t) = (x-x_0)^r (t-t_0)^s$	$U_k(t) = (t-t_0)^s \delta(k-r)\ where, \delta(k-r)= \begin{cases} 1, & k=r \\ 0, & else \end{cases}$
$u(x,t) = f(x,t)g(x,t)$	$U_k(t) = \sum_{r=0}^{k} F_r(t)G_{k-r}(t)$
$u(x,t) = f(x,t)g(x,t)h(x,t)$	$U_k(t) = \sum_{r=0}^{k}\sum_{q=0}^{r} F_q(t)G_{r-q}(t)H_{k-r}(t)$
$u(x,t) = sin(ax+\alpha)$	$U_k(t) = \frac{a^k}{k!}\left[sin(\frac{k\pi}{2}+\alpha)\right]$
$u(x,t) = cos(ax+\alpha)$	$U_k(t) = \frac{a^k}{k!}\left[cos(\frac{k\pi}{2}+\alpha)\right]$

In this technique, first we apply the Laplace transformation on equation (3.1), with respect to ‘t’, therefore we get

$$L\Big[\mathcal{L}[u(x,t)]\Big] = L\Big[N[u(x,t)] + R[u(x,t)]\Big] + L\Big[\phi(x,t)\Big]. \tag{3.4}$$

By using I.C. (3.2), we get

$$s^n \widetilde{u}(x,s) = \widetilde{h}(x,s) + L\Big[N[u(x,t)] + R[u(x,t)]\Big] + \widetilde{\phi}(x,s), \tag{3.5}$$

where

$$\widetilde{h}(x,s) = \sum_{r=0}^{n-1} s^{n-(r+1)} u_t^{(r)}(x,0)$$

and $\widetilde{u}(x,s)$ and $\widetilde{\phi}(x,s)$ are the Laplace transformed forms of $u(x,t)$ and $\phi(x,t)$ respectively.
Now, dividing by s^n on both sides, we get

$$\widetilde{u}(x,s) = \frac{\widetilde{h}(x,s)}{s^n} + \frac{1}{s^n}L\Big[N[u(x,t)] + R[u(x,t)]\Big]\Big] + \frac{\widetilde{\phi}(x,s)}{s^n},$$

or

$$\widetilde{u}(x,s) = \widetilde{f}(x,s) + \frac{1}{s^n}L\Big[N[u(x,t)] + R[u(x,t)]\Big], \tag{3.6}$$

where

$$\widetilde{f}(x,s) = \frac{\widetilde{h}(x,s)}{s^n} + \frac{\widetilde{\phi}(x,s)}{s^n}.$$

In the second step, we apply inverse Laplace transform on the equation (3.6) with respect to ‘s’, and we get

$$L^{-1}\big[\widetilde{u}(x,s)\big] = L^{-1}\Big[\widetilde{f}(x,s)\Big] + L^{-1}\Big[\frac{1}{s^n}L\big[N[u(x,t)] + R[u(x,t)]\big]\Big]$$

or

$$u(x,t) = f(x,t) + L^{-1}\left[\frac{1}{s^n}L\big[N[u(x,t)] + R[u(x,t)]\big]\right]. \tag{3.7}$$

Now, we apply differential transform method on equations (3.7) and (3.3) with respect to 'x', and we get

$$U_k(t) = F_k(t) + L^{-1}\left[\frac{1}{s^n}L\left[N[\sum_{r=0}^{k} U_r(t)U_{k-r}(t)] + R[U_k(t)]\right]\right] \tag{3.8}$$

and

$$U_0(t) = h_0(t), \;\; U_1(t) = h_1(t), \tag{3.9}$$

where $U_k(t)$ and $F_k(t)$ are the differential transform of $u(x,t)$ and $f(x,t)$ respectively.
By the above recurrence equation (3.8) and the initial conditions (3.9), the closed form of the solution can be written as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x)^k. \tag{3.10}$$

4 NUMERICAL APPLICATIONS

In this section, we apply Differential Transform Method (DTM) coupled with Laplace transform to solve linear & nonlinear Wave Equations [9]. Numerical results are very encouraging.

Example 4.1: Wave-like equation (see Fig. 1.)
Consider the following homogeneous wave like equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

with the initial conditions,

$$u(x,0) = 0, \quad u_t(x,0) = x^2, \tag{4.2}$$

and

$$u(0,t) = 0, \quad u_x(0,t) = 0. \tag{4.3}$$

In this technique, first we apply the Laplace transformation on equation (4.1) with respect to 't', therefore, we get

$$s^2L\big[u(x,t)\big] - su(x,0) - u_t(x,0) = L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

By using initial conditions from equation (4.2), we get

$$L\big[u(x,t)\big] = \frac{x^2}{s^2} + \frac{1}{s^2}L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

Now, we apply the Inverse Laplace transformation w.r.t.'s' on both sides:

$$u(x,t) = x^2t + L^{-1}\left[\frac{1}{s^2}L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right]\right]. \tag{4.4}$$

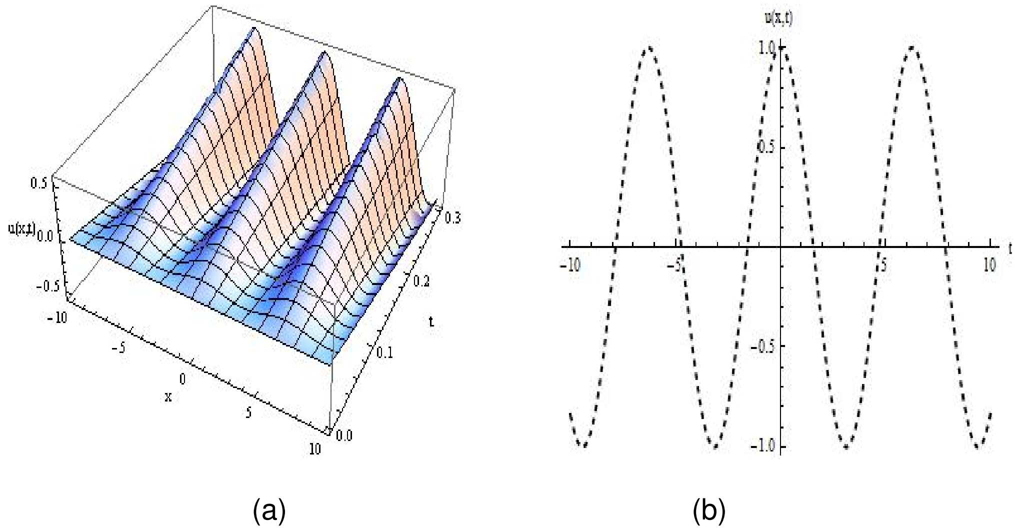


Fig. 1. (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.1.

The next step is applying the Differential transformation method on equations (4.3) and (4.4) with respect to space variable 'x', we get

$$U_k(t) = t\delta(k-2,t) + L^{-1}\left[\frac{1}{2s^2}L\left[\sum_{r=0}^{k}(r+2)(r+1)U_{r+2}(t)\delta(k-r-2,t)\right]\right]; \tag{4.5}$$

$$where \quad \delta(k-2,t) = \begin{cases} 1 & k=2 \\ 0 & otherwise \end{cases}$$

and

$$U_0(t) = 0, \quad U_1(t) = 0. \tag{4.6}$$

Substituting (4.6) into (4.5) and the following approximations are obtained successively

$$U_2(t) = sinh(t), \;\; U_3(t) = 0, \;\; U_4(t) = 0, \;\; U_5(t) = 0...$$

Finally, the closed form solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k = x^2 sinh(t).$$

which is the exact solution.

Example 4.2: Inhomogeneous wave equation (see Fig. 2.)
Consider the following inhomogeneous nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u + u^2 - xt - x^2t^2, \tag{4.7}$$

with the initial conditions,

$$u(x,0) = 0, \quad u_t(x,0) = x, \tag{4.8}$$

and

$$u(0,t) = 0, \quad u_x(0,t) = t. \tag{4.9}$$

In this technique, first we apply the Laplace transformation on equation (4.7) with respect to 't', therefore, we get

$$s^2 L\big[u(x,t)\big] - su(x,0) - u_t(x,0) = -\frac{x}{s^2} - \frac{2x^2}{s^3} + L\left[\frac{\partial^2 u}{\partial x^2} + u + u^2\right].$$

By using initial conditions from equation (4.8), we get

$$L\big[u(x,t)\big] = \frac{x}{s^2} - \frac{x}{s^4} - \frac{2x^2}{s^5} + \frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + u + u^2\right].$$

Now, we apply the Inverse Laplace transformation w.r.t.'s' on both sides:

$$u(x,t) = xt - \frac{xt^3}{3!} - \frac{2x^2 t^4}{4!} + L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + u + u^2\right]\right]. \tag{4.10}$$

The next step is applying the Differential transformation method on equations (4.9) and (4.10) with respect to space variable 'x', we get

$$\begin{aligned} U_k(t) = t\delta(k-1,t) - \frac{t^3}{3!}\delta(k-1,t) - \frac{2t^4}{4!}\delta(k-2,t) + L^{-1}\left[\frac{1}{s^2} L\Big[U_k(t)\Big]\right] \\ + L^{-1}\left[\frac{1}{s^2} L\left[(k+2)(k+1)U_{k+2}(t) + \sum_{r=0}^{k} U_r(t)U_{k-r}(t)\right]\right]; \end{aligned} \tag{4.11}$$

where

$$\delta(k-1,t) = \begin{cases} 1 & k=1 \\ 0 & otherwise \end{cases}$$

and

$$U_0(t) = 0, \quad U_1(t) = t. \tag{4.12}$$

Substituting (4.12) into (4.11) and the following approximations are obtained successively

$$U_2(t) = 0, \;\; U_3(t) = 0, \;\; U_4(t) = 0, \;\; U_5(t) = 0...$$

Finally, the closed form solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t) x^k = xt.$$

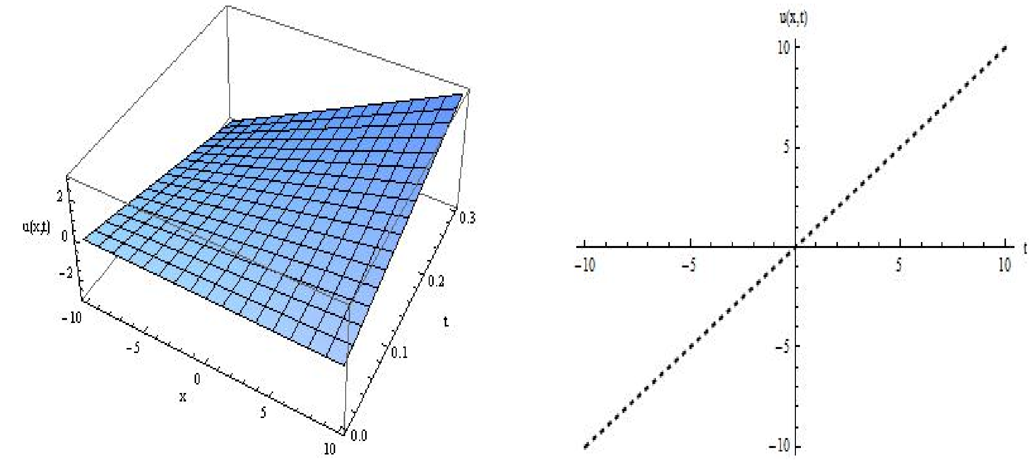
which is the exact solution.

Example 4.3: Homogeneous wave equation (see Fig. 3.)
Consider the following homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 3u, \tag{4.13}$$

with the initial conditions,

$$u(x,0) = 0, \quad u_t(x,0) = 2cos(x), \tag{4.14}$$



(a) (b)

Fig. 2. (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.2.

and

$$u(0,t) = sin(2t), \quad u_x(0,t) = 0. \tag{4.15}$$

In this technique, first we apply the Laplace transformation on equation (4.13) with respect to 't', therefore, we get

$$s^2 L\big[u(x,t)\big] - su(x,0) - u_t(x,0) = L\bigg[\frac{\partial^2 u}{\partial x^2} - 3u\bigg].$$

By using initial conditions from equation (4.14), we get

$$L\big[u(x,t)\big] = \frac{2cos(x)}{s^2} + \frac{1}{s^2} L\bigg[\frac{\partial^2 u}{\partial x^2} - 3u\bigg].$$

Now, we apply the Inverse Laplace transformation w.r.t.'s' on both sides:

$$u(x,t) = 2tcos(x) + L^{-1}\bigg[\frac{1}{s^2} L\bigg[\frac{\partial^2 u}{\partial x^2} - 3u\bigg]\bigg]. \tag{4.16}$$

The next step is applying the Differential transformation method on equations (4.15) and (4.16) with respect to space variable 'x', we get

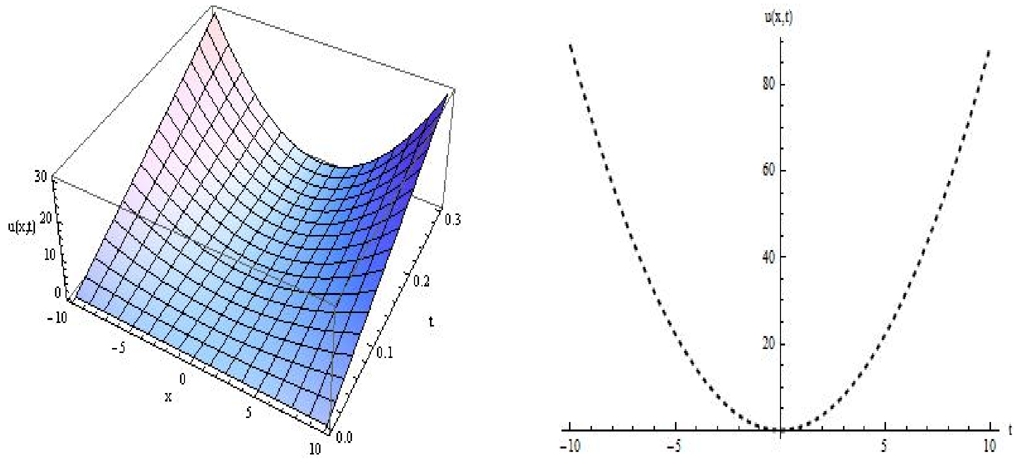
$$U_k(t) = \frac{2t}{k!} cos(\frac{k\pi}{2}) + L^{-1}\bigg[\frac{1}{s^2} L\bigg[(k+2)(k+1)U_{k+2}(t) - 3U_k(t)\bigg]\bigg], \tag{4.17}$$

and

$$U_0(t) = sin(2t), \quad U_1(t) = 0. \tag{4.18}$$

Substituting (4.18) into (4.17) and the following approximations are obtained successively

$$U_2(t) = -\frac{1}{2} sin(2t), \;\; U_3(t) = 0, \;\; U_4(t) = \frac{1}{4!} sin(2t), \;\; U_5(t) = 0...$$



(a) (b)

Fig. 3. (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.3.

Finally, the solution in series form is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k = sin(2t)\Big[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\Big].$$

the closed form solution is given as

$$u(x,t) = sin(2t)cos(x).$$

which is the exact solution.

Example 4.4: Wave-like equation in unbounded domain (see Fig. 4.)
Consider the following homogeneous wave equation in an unbounded domain

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{4.19}$$

with the initial conditions,

$$u(x,0) = sin(x), \quad u_t(x,0) = 0, \tag{4.20}$$

and

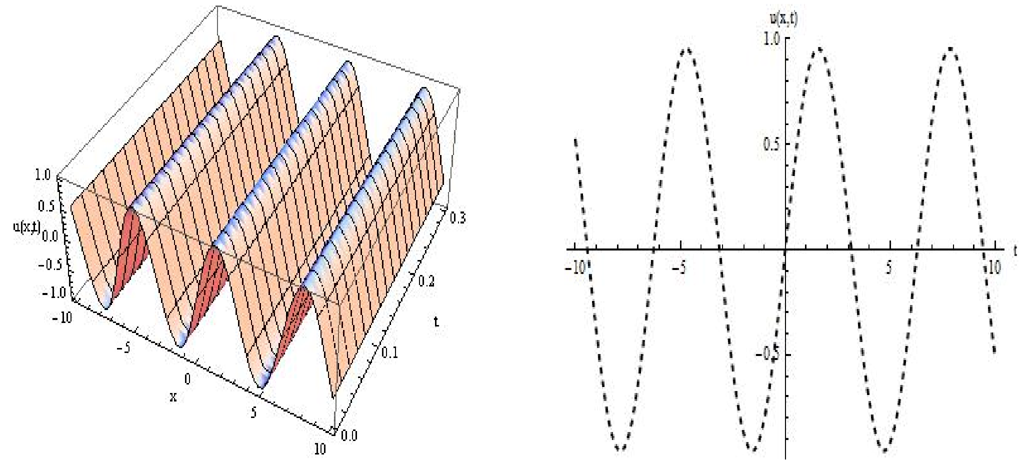
$$u(0,t) = 0, \quad u_x(0,t) = cos(t). \tag{4.21}$$

In this technique, first we apply the Laplace transformation on equation (4.19) with respect to 't', therefore, we get

$$s^2 L\big[u(x,t)\big] - su(x,0) - u_t(x,0) = L\Big[\frac{\partial^2 u}{\partial x^2}\Big].$$

By using initial conditions from equation (4.20), we get

$$L\big[u(x,t)\big] = \frac{sin(x)}{s} + \frac{1}{s^2}L\Big[\frac{\partial^2 u}{\partial x^2}\Big].$$



(a) (b)

Fig. 4. (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.4.

Now, we apply the Inverse Laplace transformation w.r.t. ‘s’ on both sides:

$$u(x,t) = sin(x) + L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u}{\partial x^2}\right]\right]. \tag{4.22}$$

The next step is applying the Differential transformation method on equations (4.21) and (4.22) with respect to space variable ‘x’, we get

$$U_k(t) = \frac{1}{k!}sin(\frac{k\pi}{2}) + L^{-1}\left[\frac{1}{s^2}L\Big[(k+2)(k+1)U_{k+2}(t)\Big]\right], \tag{4.23}$$

and

$$U_0(t) = 0, \quad U_1(t) = cos(t). \tag{4.24}$$

Substituting (4.24) into (4.23) and the following approximations are obtained successively

$$U_2(t) = 0, \;\; U_3(t) = -\frac{1}{3!}cos(t), \;\; U_4(t) = 0, \;\; U_5(t) = \frac{1}{5!}cos(t)...$$

Finally, the solution in series form is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k = cos(t)\Big[x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\Big].$$

the closed form solution is given as

$$u(x,t) = sin(x)cos(t).$$

which is the exact solution.

5 CONCLUSION

In this study, the coupling of Differential Transform Method (DTM) and Laplace Transformation is successfully expanded for the solution of linear & nonlinear Wave equations. The proposed algorithm is suitable for such problems and it gives rapidly converging series solutions. The Laplace differential transform method is an effective and convenient method to handle these type of physical problems. The present method reduces the computational work and subsequent results are fully supportive of the reliability and efficiency of the suggested scheme. The computations of this paper have been carried out using Mathematica 8.

COMPETING INTERESTS

The authors declare that no competing interests exist.

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