

Polyadic Cyclic Codes over a Non-Chain Ring

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How to cite this paper: Goyal, M. and Raka, M. (2021) Polyadic Cyclic Codes over a Non-Chain Ring. *Journal of Computer and Communications*, 9, 36-57.

<https://doi.org/10.4236/jcc.2021.95004>

Received: March 1, 2021

Accepted: May 17, 2021

Published: May 20, 2021

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Abstract

Let $f(u)$ and $g(v)$ be two polynomials of degree k and ℓ respectively, not both linear which split into distinct linear factors over \mathbb{F}_q . Let

$\mathcal{R} = \mathbb{F}_q[u, v] / \langle f(u), g(v), uv - vu \rangle$ be a finite commutative non-chain ring.

In this paper, we study polyadic codes and their extensions over the ring \mathcal{R} . We give examples of some polyadic codes which are optimal with respect to Griesmer type bound for rings. A Gray map is defined from $\mathcal{R}^n \rightarrow \mathbb{F}_q^{k\ell n}$ which preserves duality. The Gray images of polyadic codes and their extensions over the ring \mathcal{R} lead to construction of self-dual, isodual, self-orthogonal and complementary dual (LCD) codes over \mathbb{F}_q . Some examples are also given to illustrate this.

Keywords

Polyadic Codes and Their Extensions, Griesmer Bound, Gray Map, Self-Dual and Self-Orthogonal Codes, Isodual Codes, LCD Codes

1. Introduction

Polyadic cyclic codes or simply called polyadic codes form an important class of cyclic codes. They have rich algebraic structures for efficient error detection and correction, which explains their preferred role in engineering. Polyadic codes generalize quadratic residue codes, duadic codes, triadic codes and m -adic residue codes.

Codes over finite rings have been known for several decades, but interest in these codes increased substantially after a break-through work by Hammons *et al.* in 1994, which shows that some well known binary non-linear codes like Kerdock codes and Preparata codes can be constructed from linear codes over \mathbb{Z}_4 . Since then, a lot of research has been done on cyclic codes, in particular on quadratic residue codes over different types of finite rings such as integer residue

rings \mathbb{Z}_m , Galois rings $GR(p^s, m)$, chain rings and non-chain rings. Kaya *et al.* [1] and Zhang *et al.* [2] studied quadratic residue codes over a non-chain ring $\mathbb{F}_p + v\mathbb{F}_p$, where $v^2 = v$ and p is an odd prime. Bayram and Siap [3] considered cyclic and constacyclic codes over $\mathbb{F}_p[v]/\langle v^p - v \rangle$, where p is a prime. Kaya *et al.* [4] studied quadratic residue codes over $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$, whereas Liu *et al.* [5] studied them over non-local ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$, where $u^3 = u$ and p is an odd prime. The authors [6] along with Kathuria extended their results over the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p + u^3\mathbb{F}_p$, where $u^4 = u$ and $p \equiv 1 \pmod{3}$. In [7], the authors studied quadratic residue codes and their extensions over the ring $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p + \dots + u^{m-1}\mathbb{F}_p$, where $u^m = u$, m any integer greater than 1 and p is a prime satisfying $p \equiv 1 \pmod{m-1}$. In [8], the authors studied duadic codes over the ring $\mathbb{F}_q[u]/\langle u^m - u \rangle$, where q is a prime power satisfying $q \equiv 1 \pmod{m-1}$. In [9], the authors considered a more general non-chain ring $\mathbb{F}_q[u]/\langle f(u) \rangle$, where q is a prime power and $f(u)$ is a polynomial of degree $m \geq 2$, which splits into distinct linear factors over \mathbb{F}_q and studied duadic and triadic codes over it generalizing all the previous results. In another paper [10], the authors have studied duadic negacyclic codes over the ring $\mathbb{F}_q[u]/\langle f(u) \rangle$. In [11], Kuruz *et al.* studied m -adic residue codes over $\mathbb{F}_q[v]/\langle v^2 - v \rangle$.

Recently people have started studying codes over finite commutative non-chain rings having 2 or more variables. Ashraf and Mohammad [12] studied cyclic codes over $\mathbb{F}_p[u, v]/\langle u^2 - 1, v^3 - v, uv - vu \rangle$. They [13] also studied skew-cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, where $u^2 = u, v^2 = v, uv = vu = 0$. Srinivasulu and Bhaintwal [14] studied linear codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, where $u^2 = 0, v^2 = v, uv = vu$, a non-chain extension of $\mathbb{F}_2 + u\mathbb{F}_2$. Yao, Shi and Solé [15] studied skew cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v, uv = vu$ and q is a prime power. Islam and Prakash [16] studied skew cyclic and skew constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v$ and $uv = vu$. Note that all the polynomials considered namely $u^2 - u, u^3 - u, v^p - v$ or $u^m - u$ with $q \equiv 1 \pmod{m-1}$ etc. split into distinct linear factors over \mathbb{F}_q .

In this paper, we study cyclic and polyadic cyclic codes over a more general ring. Let $f(u)$ and $g(v)$ be two non-constant polynomials of degree k and ℓ respectively (k and ℓ not both 1), which split into distinct linear factors over \mathbb{F}_q . Let $\mathcal{R} = \mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu \rangle$ be a finite commutative non-chain ring. Here we discuss cyclic codes and their duals over the ring \mathcal{R} , define polyadic codes over \mathcal{R} in terms of idempotent generators and study some of their properties. We also give examples of some codes that have optimal parameters with respect to Griesmer type bound for rings. A Gray map is defined from $\mathcal{R}^n \rightarrow \mathbb{F}_q^{k\ell n}$ which preserves linearity and in some special case preserves duality. The Gray images of polyadic codes over the ring \mathcal{R} and their extensions lead to construction of self-dual, isodual, self-orthogonal and complementary dual (LCD) codes over \mathbb{F}_q .

The paper is organized as follows: In Section 2, we give some preliminaries in-

cluding Griesmer type bound for codes over rings, recall polyadic codes of length n over \mathbb{F}_q and give some of their properties. In Section 3, we study the ring \mathcal{R} , cyclic codes over ring \mathcal{R} and define the Gray map $\Phi: \mathcal{R}^n \rightarrow \mathbb{F}_q^{kln}$. In Section 4, we study polyadic codes over \mathcal{R} , their extensions, their Gray images and discuss Griesmer type bound for these codes. We give some examples to illustrate our theory.

2. Preliminaries

A cyclic code \mathcal{C} of length n over \mathbb{F}_q can be regarded as an ideal of the ring $\mathbb{S}_n = \mathbb{F}_q[x]/\langle x^n - 1 \rangle$. It has a unique generating polynomial $g(x)$ and a unique idempotent generator $e(x)$. The set $\{i: \alpha^i \text{ is a zero of } g(x)\}$, where α is a primitive n th root of unity in some extension field of \mathbb{F}_q , is called the defining set of \mathcal{C} .

A polynomial $a(x) = \sum_i a_i x^i \in \mathbb{S}_n$ is called even-like if $a(1) = 0$ otherwise it is called odd-like. A code \mathcal{C} is called even-like if all its codewords are even-like otherwise it is called odd-like.

For $(a, n) = 1$, $\mu_a: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ defined as $\mu_a(i) = ai \pmod{n}$ is called a multiplier, where $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$. It is extended on \mathbb{S}_n by defining $\mu_a(\sum_i f_i x^i) = \sum_i f_i x^{\mu_a(i)}$.

For a linear code \mathcal{C} over \mathbb{F}_q , the dual code \mathcal{C}^\perp is defined as $\mathcal{C}^\perp = \{x \in \mathbb{F}_q^n \mid x \cdot y = 0 \text{ for all } y \in \mathcal{C}\}$, where $x \cdot y$ denotes the usual Euclidean inner product. \mathcal{C} is self-dual if $\mathcal{C} = \mathcal{C}^\perp$ and self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^\perp$. A code \mathcal{C} is called isodual if it is equivalent to its dual \mathcal{C}^\perp . A linear code \mathcal{C} whose dual \mathcal{C}^\perp satisfies $\mathcal{C} \cap \mathcal{C}^\perp = \{0\}$ is called a complementary dual (LCD) code.

Let $\bar{j}(x) = \frac{1}{n}(1 + x + x^2 + \dots + x^{n-1})$. The even weight $[n, n-1, 2]$ cyclic code \mathbb{E}_n over \mathbb{F}_q has generating idempotent $1 - \bar{j}(x)$, its dual is the repetition code $[n, 1, n]$ with generating idempotent $\bar{j}(x)$.

The following is a well known result, see [17]:

Lemma 1: 1) Let \mathcal{C} be a cyclic code of length n over a finite field \mathbb{F}_q with defining set T . Then the defining set of $\mu_a(\mathcal{C})$ is $\mu_{a^{-1}}(T)$ and that of \mathcal{C}^\perp is $\mathbb{Z}_n - \mu_{-1}(T)$.

2) Let \mathcal{C} and \mathcal{D} be cyclic codes of length n over a finite field \mathbb{F}_q with defining sets T_1 and T_2 respectively. Then $\mathcal{C} \cap \mathcal{D}$ and $\mathcal{C} + \mathcal{D}$ are cyclic codes with defining sets $T_1 \cup T_2$ and $T_1 \cap T_2$ respectively.

3) Let \mathcal{C} and \mathcal{D} be cyclic codes of length n over \mathbb{F}_q generated by the idempotents E_1, E_2 in $\mathbb{F}_q[x]/\langle x^n - 1 \rangle$, then $\mathcal{C} \cap \mathcal{D}$ and $\mathcal{C} + \mathcal{D}$ are generated by the idempotents $E_1 E_2$ and $E_1 + E_2 - E_1 E_2$ respectively.

4) Let \mathcal{C} be a cyclic code of length n over \mathbb{F}_q generated by the idempotent E , then $\mu_a(\mathcal{C})$ is generated by $\mu_a(E)$ and \mathcal{C}^\perp is generated by the idempotent $1 - E(x^{-1})$.

A linear code \mathcal{C} over a finite commutative ring R is an R -submodule of R^n . Dual of a linear code over a finite commutative ring is defined in the same way and results in Lemma 1 (3) and (4) also hold true over any finite ring.

2.1. Griesmer Type Bound for Codes over Rings

Let R be a finite commutative quasi-Frobenious ring. For a linear code C over R , the value $k(C)$ is defined as the rank of minimal free R -submodules of R^n which contain C . Let $R = \bigoplus_{\alpha \in \Lambda} Re_\alpha$, where e_α are central orthogonal idempotents with, $1_R = \sum_{\alpha \in \Lambda} e_\alpha$. Then $R_\alpha = Re_\alpha$ is also a QF ring for each $\alpha \in \Lambda$. Let $J(R)$ denote the Jacobson radical of R . If C is a linear code of length n over R , then $C_\alpha = Ce_\alpha$ is a linear code of length n over R_α .

The following Griesmer type bound is due to Shiromoto and Storme ([18], Theorem 2.6).

Theorem 1: Let $R = \bigoplus_{\alpha \in \Lambda} Re_\alpha$ be a finite quasi-Frobenious ring such that R_α is a local ring for all $\alpha \in \Lambda$ and let q_α be the prime power such that $|R_\alpha/J(R_\alpha)| = q_\alpha$ for each $\alpha \in \Lambda$. If C is a linear code of length n over R , then

$$n \geq \sum_{i=0}^{k(C)-1} \left\lceil \frac{d(C)}{q^i} \right\rceil \quad (1)$$

where $q = \max_{\alpha \in \Lambda} \{q_\alpha\}$, $k(C) = \max_{\alpha \in \Lambda} \{k(C_\alpha)\}$ and $d(C) = \min_{\alpha \in \Lambda} \{d(C_\alpha)\}$.

The code C over R is said to have parameters $[n, k(C), d(C)]$.

2.2. Polyadic Cyclic Codes over \mathbb{F}_q

Let $(n, q) = 1$ and suppose

$$\mathbb{Z}_n = S_1 \cup S_2 \cup \dots \cup S_m \cup S_\infty, \quad (2)$$

where

- 1) S_1, S_2, \dots, S_m and S_∞ are union of q -cyclotomic cosets mod n ,
- 2) S_1, S_2, \dots, S_m and S_∞ are pairwise disjoint,
- 3) there exists a multiplier μ_a , $(a, n) = 1$ such that $\mu_a(S_i) = S_{i+1}$, for $1 \leq i \leq m$, the subscripts are taken modulo m and $\mu_a(S_\infty) = S_\infty$.

It is clear that $0 \in S_\infty$ always. Let $S'_\infty = S_\infty - \{0\}$.

Then codes, for $1 \leq i \leq m$, having $S_i \cup S'_\infty$ or $(S_i \cup S_\infty)^c$ as their defining sets are called odd-like polyadic codes and the codes having $(S_i \cup S'_\infty)^c$ or $S_i \cup S_\infty$ as their defining sets are the associated even-like polyadic codes. Let \mathbb{D}_i denote the odd-like codes having $S_i \cup S'_\infty$ as their defining sets; \mathbb{D}'_i denote the odd-like codes having $(S_i \cup S_\infty)^c$ as their defining sets; \mathbb{C}_i denote the even-like codes having $(S_i \cup S'_\infty)^c$ as their defining sets; and \mathbb{C}'_i denote the even-like codes having $S_i \cup S_\infty$ as their defining sets.

In the special case, when $m = 2$ and $S_\infty = \{0\}$, polyadic codes are duadic codes [19]. When $m = 3$, polyadic codes are triadic codes as defined by Pless and Rushanan [20]. When $n = p$, an odd prime, $m \mid (p-1)$, $S_\infty = \{0\}$, $\mathbb{Z}_p^* = \langle b \rangle$, $S_1 = \left\{ b^{mr}, 1 \leq r \leq \frac{p-1}{m} \right\}$, $S_i = b^{i-1}S_1$, then polyadic codes are m -adic residue codes as defined by Job [21]. A polyadic code of prime length p exists if and only if q is an m -adic residue mod p , see Brualdi and Pless [22]. When n is a prime power, the conditions for the existence of polyadic codes over \mathbb{F}_q were obtained

by Sharma *et al.* [23] and for general n see Bakshi *et al.* [24].

Clearly $\mathbb{D}_1, \mathbb{D}_2, \dots, \mathbb{D}_m$ are equivalent codes; $\mathbb{D}'_1, \mathbb{D}'_2, \dots, \mathbb{D}'_m$ are equivalent; $\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_m$ are equivalent; and $\mathbb{C}'_1, \mathbb{C}'_2, \dots, \mathbb{C}'_m$ are equivalent codes.

For $1 \leq i \leq m$, let $e_i(x)$ and $e'_i(x)$ be the even-like idempotent generators of even-like polyadic codes \mathbb{C}_i and \mathbb{C}'_i respectively, $d_i(x)$ and $d'_i(x)$ be odd-like idempotent generators of odd-like polyadic codes \mathbb{D}_i and \mathbb{D}'_i respectively.

As the defining set of \mathbb{C}_1 is $S_2 \cup S_3 \cup \dots \cup S_m \cup \{0\}$, the defining set of $\mu_a(\mathbb{C}_1)$ is $\mu_{a-1}(S_2 \cup S_3 \cup \dots \cup S_m \cup \{0\}) = S_1 \cup S_2 \cup \dots \cup S_{m-1} \cup \{0\}$. Therefore $\mu_a(\mathbb{C}_1) = \mathbb{C}_m$ and hence $\mu_a(e_1) = e_m$. Similarly, $\mu_a(e_i) = e_{i-1}$ for $1 \leq i \leq m$ and $\mu_a(d_i) = d_{i-1}$ for $1 \leq i \leq m$. Similar results hold for e'_i and d'_i .

Let the set $\{1, 2, \dots, m\}$ be denoted by A . Similar to the properties of triadic codes obtained in [9], we have the following results for polyadic codes over \mathbb{F}_q .

Proposition 1: For any subset $\{t_1, t_2, \dots, t_r\} \subseteq A$, where $2 \leq r \leq m$, we have

- 1) $\mathbb{C}_1 \cap \mathbb{C}_2 \cap \dots \cap \mathbb{C}_m = \mathbb{C}_{t_1} \cap \mathbb{C}_{t_2} \cap \dots \cap \mathbb{C}_{t_r}$,
- 2) $\mathbb{C}_1 + \mathbb{C}_2 + \dots + \mathbb{C}_m = \mathbb{E}_n = \langle x-1 \rangle = \langle 1 - \bar{j}(x) \rangle$,
- 3) $\mathbb{D}_1 + \mathbb{D}_2 + \dots + \mathbb{D}_m = \mathbb{D}_{t_1} + \mathbb{D}_{t_2} + \dots + \mathbb{D}_{t_r}$,
- 4) $\mathbb{D}_1 \cap \mathbb{D}_2 \cap \dots \cap \mathbb{D}_m = \langle \bar{j}(x) \rangle$,
- 5) $\mathbb{C}_i + \mathbb{D}_i = \mathbb{S}_n$, $\mathbb{C}_i \cap \mathbb{D}_i = \{0\}$ for $1 \leq i \leq m$,
- 6) $e_1(x)e_2(x)\dots e_m(x) = e_{t_1}(x)e_{t_2}(x)\dots e_{t_r}(x)$,
- 7) $e_1(x) + e_2(x) + \dots + e_m(x) - (m-1)e_1(x)e_2(x)\dots e_m(x) = 1 - \bar{j}(x)$,
- 8)

$$\sum_{i=1}^m d_i(x) - \sum_{i < j} d_i(x)d_j(x) + \sum_{i < j < k} d_i(x)d_j(x)d_k(x) - \dots (-1)^{m-1} \prod_{i=1}^m d_i(x) \\ = \sum_{i=1}^r d_{t_i}(x) - \sum_{t_i < t_j} d_{t_i}(x)d_{t_j}(x) + \sum_{t_i < t_j < t_k} d_{t_i}(x)d_{t_j}(x)d_{t_k}(x) - \dots (-1)^{r-1} \prod_{i=1}^r d_{t_i}(x)$$

- 9) $d_1(x)d_2(x)\dots d_m(x) = \bar{j}(x)$,
- 10) $d_i(x) = 1 - e_i(x)$, $e_i(x)d_i(x) = 0$ for $1 \leq i \leq m$.

Proof: By Lemma 1 (2), the defining set of each of $\mathbb{C}_1 \cap \mathbb{C}_2 \cap \dots \cap \mathbb{C}_m$, $\mathbb{C}_{t_1} \cap \mathbb{C}_{t_2} \cap \dots \cap \mathbb{C}_{t_r}$ is $S_1 \cup S_2 \cup \dots \cup S_m \cup \{0\}$, hence they are equal. The defining set of $\mathbb{C}_1 + \mathbb{C}_2 + \dots + \mathbb{C}_m$ is $\{0\}$, which is the defining set of even weight code \mathbb{E}_n having generating idempotent $1 - \bar{j}(x)$. Again by Lemma 1 (2), the defining set of each of $\mathbb{D}_1 + \mathbb{D}_2 + \dots + \mathbb{D}_m$, $\mathbb{D}_{t_1} + \mathbb{D}_{t_2} + \dots + \mathbb{D}_{t_r}$ is S'_∞ , hence they are all equal. The defining set of $\mathbb{D}_1 \cap \mathbb{D}_2 \cap \dots \cap \mathbb{D}_m$ is $\mathbb{Z}_n - \{0\}$, which is the defining set of the repetition code having generating idempotent $\bar{j}(x)$. The defining set of $\mathbb{C}_i \cap \mathbb{D}_i$ is whole of \mathbb{Z}_n , hence it is $\{0\}$ in the ring $\mathbb{S}_n = \mathbb{F}_q[x]/\langle x^n - 1 \rangle$; whereas defining set of $\mathbb{C}_i + \mathbb{D}_i$ is \emptyset , so it is $\mathbb{S}_n = \langle 1 \rangle$. The other results follow by Lemma 1 (3). \square

Proposition 2: For any subset $\{t_1, t_2, \dots, t_r\} \subseteq A$, where $2 \leq r \leq m$, we have

- 1) $\mathbb{C}'_1 \cap \mathbb{C}'_2 \cap \dots \cap \mathbb{C}'_m = \{0\}$,
- 2) $\mathbb{C}'_1 + \mathbb{C}'_2 + \dots + \mathbb{C}'_m = \mathbb{C}'_{t_1} + \mathbb{C}'_{t_2} + \dots + \mathbb{C}'_{t_r}$,
- 3) $\mathbb{D}'_1 \cap \mathbb{D}'_2 \cap \dots \cap \mathbb{D}'_m = \mathbb{D}'_{t_1} \cap \mathbb{D}'_{t_2} \dots \cap \mathbb{D}'_{t_r}$,
- 4) $\mathbb{D}'_1 + \mathbb{D}'_2 + \dots + \mathbb{D}'_m = \mathbb{S}_n = \langle 1 \rangle$,

- 5) $\mathbb{C}'_i + \mathbb{D}'_i = \mathbb{S}_n$, $\mathbb{C}'_i \cap \mathbb{D}'_i = \{0\}$ for $1 \leq i \leq m$,

$$\sum_{i=1}^m e'_i(x) - \sum_{i < j} e'_i(x)e'_j(x) + \sum_{i < j < k} e'_i(x)e'_j(x)e'_k(x) - \dots (-1)^{m-1} \prod_{i=1}^m e'_i(x)$$
- 6)
$$= \sum_{i=1}^r e'_{t_i}(x) - \sum_{t_i < t_j} e'_{t_i}(x)e'_{t_j}(x) + \sum_{t_i < t_j < t_k} e'_{t_i}(x)e'_{t_j}(x)e'_{t_k}(x) - \dots (-1)^{r-1} \prod_{i=1}^r e'_{t_i}(x)$$
- 7) $d'_1(x)d'_2(x) \cdots d'_m(x) = d'_{t_1}(x)d'_{t_2}(x) \cdots d'_{t_r}(x)$,
- 8) $d'_1(x) + d'_2(x) + \dots + d'_m(x) - (m-1)d'_1(x)d'_2(x) \cdots d'_m(x) = 1$,
- 9) $e'_1(x)e'_2(x) \cdots e'_m(x) = 0$,
- 10) $d'_i(x) = 1 - e'_i(x)$, $e'_i(x)d'_i(x) = 0$,
- 11) $\mathbb{C}_i + \langle \bar{j}(x) \rangle = \mathbb{D}'_i$, $\mathbb{C}_i \cap \langle \bar{j}(x) \rangle = \{0\}$,
- 12) $\mathbb{C}'_i + \langle \bar{j}(x) \rangle = \mathbb{D}_i$, $\mathbb{C}'_i \cap \langle \bar{j}(x) \rangle = \{0\}$,
- 13) $\mathbb{C}_i \cap \mathbb{C}'_i = \{0\}$, $\mathbb{C}_i + \mathbb{C}'_i = \langle 1 - \bar{j}(x) \rangle$,
- 14) $\mathbb{D}_i \cap \mathbb{D}'_i = \langle \bar{j}(x) \rangle$, $\mathbb{D}_i + \mathbb{D}'_i = \mathbb{S}_n$,
- 15) $e_i + \bar{j}(x) = d'_i$, $e'_i + \bar{j}(x) = d_i$, $e_i \bar{j}(x) = 0$, $e'_i \bar{j}(x) = 0$,
- 16) $e_i e'_i = 0$, $e_i + e'_i = 1 - \bar{j}(x)$, $d_i d'_i = \bar{j}(x)$, $d_i + d'_i = 1 + \bar{j}(x)$.

Proof: Statements (1) to (10) are similar to those of (1) to (10) of Proposition 1. For (11), we note that the defining set of $\langle \bar{j}(x) \rangle$ is $\mathbb{Z}_n - \{0\}$. Therefore the defining set of $\mathbb{C}_i \cap \langle \bar{j}(x) \rangle$ is \mathbb{Z}_n and defining set of $\mathbb{C}_i + \langle \bar{j}(x) \rangle$ is same as that of \mathbb{D}'_i . Similarly we have (12). The defining set of $\mathbb{C}_i \cap \mathbb{C}'_i$ is \mathbb{Z}_n and that of $\mathbb{C}_i + \mathbb{C}'_i$ is $\{0\}$. The defining set of $\mathbb{D}_i \cap \mathbb{D}'_i$ is $\mathbb{Z}_n - \{0\}$ and that of $\mathbb{D}_i + \mathbb{D}'_i$ is \emptyset . Now (15) and (16) follow by Lemma 1(3). \square

Proposition 3: Suppose S'_∞ is empty, then for any subset $\{t_1, t_2, \dots, t_r\} \subseteq A$, where $2 \leq r \leq m$, we have the following additional results:

- 1) $\mathbb{C}_1 \cap \mathbb{C}_2 \cap \dots \cap \mathbb{C}_m = \mathbb{C}_{t_1} \cap \mathbb{C}_{t_2} \cap \dots \cap \mathbb{C}_{t_r} = \{0\}$,
- 2) $\mathbb{D}_1 + \mathbb{D}_2 + \dots + \mathbb{D}_m = \mathbb{D}_{t_1} + \mathbb{D}_{t_2} + \dots + \mathbb{D}_{t_r} = \mathbb{S}_n$,
- 3) $\mathbb{D}'_1 \cap \mathbb{D}'_2 \cap \dots \cap \mathbb{D}'_m = \mathbb{D}'_{t_1} \cap \mathbb{D}'_{t_2} \cap \dots \cap \mathbb{D}'_{t_r} = \langle \bar{j}(x) \rangle$,
- 4) $\mathbb{C}'_1 + \mathbb{C}'_2 + \dots + \mathbb{C}'_m = \mathbb{C}'_{t_1} + \mathbb{C}'_{t_2} + \dots + \mathbb{C}'_{t_r} = \mathbb{E}_n = \langle 1 - \bar{j}(x) \rangle$,
- 5) $e_1(x)e_2(x) \cdots e_m(x) = e_{t_1}(x)e_{t_2}(x) \cdots e_{t_r}(x) = 0$,
- 6)
$$\sum_{i=1}^m d_i(x) - \sum_{i < j} d_i(x)d_j(x) + \sum_{i < j < k} d_i(x)d_j(x)d_k(x) - \dots (-1)^{m-1} \prod_{i=1}^m d_i(x)$$

$$= \sum_{i=1}^r d_{t_i}(x) - \sum_{t_i < t_j} d_{t_i}(x)d_{t_j}(x) + \sum_{t_i < t_j < t_k} d_{t_i}(x)d_{t_j}(x)d_{t_k}(x) - \dots (-1)^{r-1} \prod_{i=1}^r d_{t_i}(x)$$

$$= 1$$
- 7) $d'_1(x)d'_2(x) \cdots d'_m(x) = d'_{t_1}(x)d'_{t_2}(x) \cdots d'_{t_r}(x) = \bar{j}(x)$,

$$\sum_{i=1}^m e'_i(x) - \sum_{i < j} e'_i(x)e'_j(x) + \sum_{i < j < k} e'_i(x)e'_j(x)e'_k(x) - \dots (-1)^{m-1} \prod_{i=1}^m e'_i(x)$$
- 8)
$$\sum_{i=1}^r e'_{t_i}(x) - \sum_{t_i < t_j} e'_{t_i}(x)e'_{t_j}(x) + \sum_{t_i < t_j < t_k} e'_{t_i}(x)e'_{t_j}(x)e'_{t_k}(x) - \dots (-1)^{r-1} \prod_{i=1}^r e'_{t_i}(x)$$

$$= 1 - \bar{j}(x)$$

Proof is straightforward.

Proposition 4: Let \mathbb{C}_i , \mathbb{C}'_i , for $1 \leq i \leq m$, be two pairs of even-like polyadic codes over \mathbb{F}_q with \mathbb{D}_i , \mathbb{D}'_i the associated pairs of odd-like polyadic codes.

Then $\mathbb{C}_i^\perp = \mu_{-1}(\mathbb{D}_i)$ and $\mathbb{C}'_i{}^\perp = \mu_{-1}(\mathbb{D}'_i)$.

Further if $\mu_{-1}(\mathbb{D}_i) = \mathbb{D}_i$, then $\mathbb{C}_i^\perp = \mathbb{D}_i$, $\mathbb{C}'_i{}^\perp = \mathbb{D}'_i$ and so \mathbb{C}_i , \mathbb{C}'_i , \mathbb{D}_i and \mathbb{D}'_i are LCD codes.

Proof: As the defining set of \mathbb{C}_1 is $(S_1 \cup S'_\infty)^c = \{0\} \cup S_2 \cup S_3 \cup \dots \cup S_m$, the defining set of \mathbb{C}_1^\perp , by Lemma 1 (1) is

$$\begin{aligned} &= \mathbb{Z}_n - \mu_{-1}(\{0\} \cup S_2 \cup S_3 \cup \dots \cup S_m) \\ &= \mu_{-1}(\mathbb{Z}_n) - \mu_{-1}(\{0\} \cup S_2 \cup S_3 \cup \dots \cup S_m) \\ &= \mu_{-1}(S_1 \cup S_2 \cup \dots \cup S_m \cup X_\infty) - \mu_{-1}(\{0\} \cup S_2 \cup S_3 \cup \dots \cup S_m) \\ &= \mu_{-1}(S_1 \cup X'_\infty) \\ &= \mu_{-1}(\text{defining set of } \mathbb{D}_1). \end{aligned}$$

This proves that $\mathbb{C}_1^\perp = \mu_{-1}(\mathbb{D}_1)$. Similar is the proof of others. When $\mu_{-1}(\mathbb{D}_i) = \mathbb{D}_i$, we get, from Propositions 1 (5) and 2 (5), that

$\mathbb{C}_i \cap \mathbb{C}_i^\perp = \mathbb{C}_i \cap \mathbb{D}_i = \{0\}$ and $\mathbb{C}'_i \cap \mathbb{C}'_i{}^\perp = \mathbb{C}'_i \cap \mathbb{D}'_i = \{0\}$; proving that \mathbb{C}_i and \mathbb{C}'_i are LCD codes. One can check that \mathbb{D}_i and \mathbb{D}'_i are also LCD codes. \square

3. Cyclic Codes over the Ring \mathcal{R} and the Gray Map

3.1. The Ring \mathcal{R}

Let q be a prime power, $q = p^s$. Throughout the paper, \mathcal{R} denotes the commutative ring $\mathbb{F}_q[u, v] / \langle f(u), g(v), uv - vu \rangle$, where $f(u)$ and $g(v)$ are two non-constant polynomials of degree k and ℓ respectively, which split into distinct linear factors over \mathbb{F}_q . We assume that k and ℓ are not both 1, otherwise $\mathcal{R} \simeq \mathbb{F}_q$. If $\ell = 1$ or $k = 1$, then the ring $\mathcal{R} = \mathbb{F}_q[u, v] / \langle f(u), g(v), uv - vu \rangle$ is isomorphic to $\mathbb{F}_q[u] / \langle f(u) \rangle$ or $\mathbb{F}_q[v] / \langle g(v) \rangle$. Cyclic, duadic and triadic codes over $\mathbb{F}_q[u] / \langle f(u) \rangle$ have been discussed by the authors in [9].

Let $f(u) = (u - \alpha_1)(u - \alpha_2) \dots (u - \alpha_k)$, with $\alpha_i \in \mathbb{F}_q$, $\alpha_i \neq \alpha_j$ and $g(v) = (v - \beta_1)(v - \beta_2) \dots (v - \beta_\ell)$, with $\beta_i \in \mathbb{F}_q$, $\beta_i \neq \beta_j$. \mathcal{R} is a non chain ring of size $q^{k\ell}$ and characteristic p .

For $k \geq 2$ and $\ell \geq 2$, let ε_i , $1 \leq i \leq k$ and γ_j , $1 \leq j \leq \ell$, be elements of the ring \mathcal{R} given by

$$\begin{aligned} \varepsilon_i = \varepsilon_i(u) &= \frac{(u - \alpha_1)(u - \alpha_2) \dots (u - \alpha_{i-1})(u - \alpha_{i+1}) \dots (u - \alpha_k)}{(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \dots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \dots (\alpha_i - \alpha_k)} \text{ and} \\ \gamma_j = \gamma_j(v) &= \frac{(v - \beta_1)(v - \beta_2) \dots (v - \beta_{j-1})(v - \beta_{j+1}) \dots (v - \beta_\ell)}{(\beta_j - \beta_1)(\beta_j - \beta_2) \dots (\beta_j - \beta_{j-1})(\beta_j - \beta_{j+1}) \dots (\beta_j - \beta_\ell)}. \end{aligned} \tag{3}$$

If $k = 1$, we define $\varepsilon_1 = 1$ and if $\ell = 1$, we take $\gamma_1 = 1$.

For $i = 1, 2, \dots, k, j = 1, 2, \dots, \ell$, define η_{ij} as follows

$$\eta_{ij} = \eta_{ij}(u, v) = \varepsilon_i(u) \gamma_j(v). \tag{4}$$

Lemma 2: We have $\eta_{ij}^2 = \eta_{ij}$, $\eta_{ij} \eta_{rs} = 0$ for $1 \leq i, r \leq k, 1 \leq j, s \leq \ell$, $(i, j) \neq (r, s)$ and $\sum_{i,j} \eta_{ij} = 1$ in \mathcal{R} , i.e., η_{ij} 's are primitive orthogonal idempotents of the ring \mathcal{R} .

Proof: Since $\varepsilon_i \varepsilon_r = 0 \pmod{f(u)}$ for $i \neq r$ and $\gamma_j \gamma_s = 0 \pmod{g(v)}$ for

$j \neq s$, $\eta_{ij}\eta_{rs} = \varepsilon_i\gamma_j\varepsilon_r\gamma_s = 0$ in \mathcal{R} . To prove $\eta_{ij}^2 = \eta_{ij}$, it is enough to prove that $\eta_{ij}(\eta_{ij} - 1) = 0$ in \mathcal{R} . For that we need to prove $(u - \alpha_r) | \eta_{ij}(u, v)(\eta_{ij}(u, v) - 1)$ for all r and $(v - \beta_s) | \eta_{ij}(u, v)(\eta_{ij}(u, v) - 1)$ for all s . If $r \neq i$, then $\eta_{ij}(\alpha_r, v) = \varepsilon_i(\alpha_r)\gamma_j(v) = 0$. If $s \neq j$, then $\eta_{ij}(u, \beta_s) = \varepsilon_i(u)\gamma_j(\beta_s) = 0$, hence $(u - \alpha_r) | \eta_{ij}(u, v)$, for $r \neq i$ and $(v - \beta_s) | \eta_{ij}(u, v)$, for $s \neq j$. One can easily check that $(u - \alpha_i) | (\eta_{ij}(u, v) - 1)$ and $(v - \beta_j) | (\eta_{ij}(u, v) - 1)$, so $\eta_{ij}(\eta_{ij} - 1) = 0$ in \mathcal{R} and hence $\eta_{ij}^2 = \eta_{ij}$ in \mathcal{R} .

Now to prove $\sum_{i,j} \eta_{ij} = 1$ in \mathcal{R} , it is sufficient to prove that $\sum_{j=1}^{\ell} \sum_{i=1}^k \eta_{ij}(u, v) \equiv 1 \pmod{(f(u), g(v))}$. This can be easily checked as $\sum_{j=1}^{\ell} \sum_{i=1}^k \eta_{ij}(\alpha_r, v) = 1$ and $\sum_{j=1}^{\ell} \sum_{i=1}^k \eta_{ij}(u, \beta_s) = 1$ for all r and s , $r = 1, 2, \dots, k, s = 1, 2, \dots, \ell$. \square

The decomposition theorem of ring theory tells us that $\mathcal{R} = \bigoplus \eta_{ij}\mathcal{R} = \bigoplus \eta_{ij}\mathbb{F}_q$.

For a linear code \mathcal{C} of length n over the ring \mathcal{R} , let for each pair (i, j) , $1 \leq i \leq k, 1 \leq j \leq \ell$, let

$$\mathcal{C}_{ij} = \left\{ x_{ij} \in \mathbb{F}_q^n : \exists x_{rs} \in \mathbb{F}_q^n, (r, s) \neq (i, j), \text{ such that } \bigoplus_{r,s} \eta_{rs} x_{rs} \in \mathcal{C} \right\}.$$

Then \mathcal{C}_{ij} are linear codes of length n over \mathbb{F}_q , $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$ and $|\mathcal{C}| = \prod_{i,j} |\mathcal{C}_{ij}|$.

The following is a simple generalization of Theorem 1 of [9].

Theorem 2: Let $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$ be a linear code of length n over \mathcal{R} . Then

1) \mathcal{C} is cyclic over \mathcal{R} if and only if $\mathcal{C}_{ij}, 1 \leq i \leq k, 1 \leq j \leq \ell$ are cyclic over \mathbb{F}_q .

2) If $\mathcal{C}_{ij} = \langle g_{ij}(x) \rangle$, $g_{ij}(x) \in \frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle}$, $g_{ij}(x) | (x^n - 1)$ then,

$$\begin{aligned} \mathcal{C} &= \langle \eta_{11}g_{11}(x), \dots, \eta_{1\ell}g_{1\ell}(x), \eta_{21}g_{21}(x), \dots, \\ &\quad \eta_{2\ell}g_{2\ell}(x), \dots, \eta_{k1}g_{k1}(x), \dots, \eta_{k\ell}g_{k\ell}(x) \rangle, \\ &= \langle g(x) \rangle \end{aligned}$$

where $g(x) = \sum_i \sum_j \eta_{ij}g_{ij}$ and $g(x) | (x^n - 1)$.

3) Further $|\mathcal{C}| = q^{k\ell n - \sum_{j=1}^{\ell} \sum_{i=1}^k \deg(g_{ij})}$.

4) Suppose that $g_{ij}(x)h_{ij}(x) = x^n - 1, 1 \leq i \leq k, 1 \leq j \leq \ell$. Let $h(x) = \bigoplus_{i,j} \eta_{ij}h_{ij}(x)$, then $g(x)h(x) = x^n - 1$.

5) $\mathcal{C}^\perp = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}^\perp$.

6) $\mathcal{C}^\perp = \langle h^\perp(x) \rangle$, $h^\perp(x) = \bigoplus_{i,j} \eta_{ij}h_{ij}^\perp(x)$, where $h_{ij}^\perp(x)$ is the reciprocal polynomial of $h_{ij}(x), 1 \leq i \leq k, 1 \leq j \leq \ell$.

7) $|\mathcal{C}^\perp| = q^{\sum_{j=1}^{\ell} \sum_{i=1}^k \deg(g_{ij})}$.

3.2. The Gray Map

Every element $r(u, v)$ of the ring $\mathcal{R} = \mathbb{F}_q[u, v] / \langle f(u), g(v), uv - vu \rangle$ can be uniquely expressed as

$$r(u, v) = \sum_{i,j} \eta_{ij} a_{ij},$$

where $a_{ij} \in \mathbb{F}_q$ for $1 \leq i \leq k, 1 \leq j \leq \ell$.

Define a Gray map $\Phi: \mathcal{R} \rightarrow \mathbb{F}_q^{k\ell}$ by

$$r(u, v) = \sum_{i,j} \eta_{ij} a_{ij} \mapsto (a_{11}, a_{12}, \dots, a_{1\ell}, a_{21}, a_{22}, \dots, a_{2\ell}, \dots, a_{k1}, a_{k2}, \dots, a_{k\ell})V,$$

where V is any nonsingular matrix over \mathbb{F}_q of order $k\ell \times k\ell$. This map can be extended from \mathcal{R}^n to $(\mathbb{F}_q^{k\ell})^n$ component wise.

Let the Gray weight of an element $r \in \mathcal{R}$ be $w_G(r) = w_H(\Phi(r))$, the Hamming weight of $\Phi(r)$. The Gray weight of a codeword

$c = (c_0, c_1, \dots, c_{n-1}) \in \mathcal{R}^n$ is defined as

$$w_G(c) = \sum_{i=0}^{n-1} w_G(c_i) = \sum_{i=0}^{n-1} w_H(\Phi(c_i)) = w_H(\Phi(c)).$$

For any two elements $c_1, c_2 \in \mathcal{R}^n$, the Gray distance d_G is given by

$$d_G(c_1, c_2) = w_G(c_1 - c_2) = w_H(\Phi(c_1) - \Phi(c_2)).$$

Theorem 3: The Gray map Φ is an \mathbb{F}_q -linear, one to one and onto map. It is also distance preserving map from $(\mathcal{R}^n, \text{Gray distance } d_G)$ to $(\mathbb{F}_q^{k\ell n}, \text{Hamming distance } d_H)$. Further if the matrix V satisfies $VV^T = \lambda I_{k\ell}$, $\lambda \in \mathbb{F}_q^*$, where V^T denotes the transpose of the matrix V , then $\Phi(\mathcal{C}^\perp) = (\Phi(\mathcal{C}))^\perp$ for any linear code \mathcal{C} over \mathcal{R} .

Proof. The first two assertions hold as V is an invertible matrix over \mathbb{F}_q .

Let now $V = (V_1^T, V_2^T, \dots, V_{k\ell}^T)$, where

$V_i = (v_{11}^{(i)}, v_{12}^{(i)}, \dots, v_{1\ell}^{(i)}, v_{21}^{(i)}, v_{22}^{(i)}, \dots, v_{2\ell}^{(i)}, \dots, v_{k1}^{(i)}, v_{k2}^{(i)}, \dots, v_{k\ell}^{(i)})$ is a $1 \times k\ell$ row vector, satisfying $VV^T = \lambda I_{k\ell}$. So that

$$\sum_{t=1}^{k\ell} (v_{rs}^{(t)})^2 = \lambda \text{ for all } 1 \leq r \leq k, 1 \leq s \leq \ell \text{ and } \sum_{t=1}^{k\ell} v_{rs}^{(t)} v_{wy}^{(t)} = 0 \text{ for } (r, s) \neq (w, y). \quad (5)$$

Let \mathcal{C} be a linear code over \mathcal{R} . Let $r = (r_0, r_1, \dots, r_{n-1}) \in \mathcal{C}^\perp$,

$s = (s_0, s_1, \dots, s_{n-1}) \in \mathcal{C}$, where $r_i = \eta_{11} a_{11}^{(i)} + \eta_{12} a_{12}^{(i)} + \dots + \eta_{k\ell} a_{k\ell}^{(i)}$ and

$s_i = \eta_{11} b_{11}^{(i)} + \eta_{12} b_{12}^{(i)} + \dots + \eta_{k\ell} b_{k\ell}^{(i)}$. So that $r \cdot s = 0$. It is enough to prove that

$\Phi(r) \cdot \Phi(s) = 0$. Using the properties of η_{ij} 's from Lemma 2, we get

$$r_i s_i = \eta_{11} a_{11}^{(i)} b_{11}^{(i)} + \eta_{12} a_{12}^{(i)} b_{12}^{(i)} + \dots + \eta_{k\ell} a_{k\ell}^{(i)} b_{k\ell}^{(i)}.$$

Then

$$0 = r \cdot s = \sum_{i=0}^{n-1} r_i s_i = \sum_{i=0}^{n-1} \sum_{r=1}^k \sum_{s=1}^\ell \eta_{rs} a_{rs}^{(i)} b_{rs}^{(i)} = \sum_{r=1}^k \sum_{s=1}^\ell \eta_{rs} \left(\sum_{i=0}^{n-1} a_{rs}^{(i)} b_{rs}^{(i)} \right)$$

implies that

$$\sum_{i=0}^{n-1} a_{rs}^{(i)} b_{rs}^{(i)} = 0, \text{ for all } r, s, 1 \leq r \leq k, 1 \leq s \leq \ell. \quad (6)$$

Now

$$\Phi(r_i) = (a_{11}^{(i)}, a_{12}^{(i)}, \dots, a_{k\ell}^{(i)})V = \left(\sum_{r=1}^k \sum_{s=1}^\ell a_{rs}^{(i)} v_{rs}^{(1)}, \sum_{r=1}^k \sum_{s=1}^\ell a_{rs}^{(i)} v_{rs}^{(2)}, \dots, \sum_{r=1}^k \sum_{s=1}^\ell a_{rs}^{(i)} v_{rs}^{(k\ell)} \right)$$

Similarly

$$\Phi(s_i) = \left(\sum_{w=1}^k \sum_{y=1}^\ell b_{wy}^{(i)} v_{wy}^{(1)}, \sum_{w=1}^k \sum_{y=1}^\ell b_{wy}^{(i)} v_{wy}^{(2)}, \dots, \sum_{w=1}^k \sum_{y=1}^\ell b_{wy}^{(i)} v_{wy}^{(k\ell)} \right).$$

Using (5) and (6), we find that

$$\begin{aligned} \Phi(r) \cdot \Phi(s) &= \sum_{i=0}^{n-1} \Phi(r_i) \cdot \Phi(s_i) = \sum_{i=0}^{n-1} \sum_{t=1}^{k\ell} \sum_{w=1}^k \sum_{y=1}^\ell \sum_{r=1}^k \sum_{s=1}^\ell a_{rs}^{(i)} b_{wy}^{(i)} v_{rs}^{(t)} v_{wy}^{(t)} \\ &= \sum_{i=0}^{n-1} \sum_{r=1}^k \sum_{s=1}^\ell a_{rs}^{(i)} b_{rs}^{(i)} \left(\sum_{t=1}^{k\ell} (v_{rs}^{(t)})^2 \right) + \sum_{i=0}^{n-1} \sum_{w=1}^k \sum_{y=1}^\ell \sum_{\substack{r=1 \\ (r,s) \neq (w,y)}}^k \sum_{s=1}^\ell a_{rs}^{(i)} b_{wy}^{(i)} \left(\sum_{t=1}^{k\ell} v_{rs}^{(t)} v_{wy}^{(t)} \right) \\ &= \lambda \sum_{i=0}^{n-1} \sum_{r=1}^k \sum_{s=1}^\ell a_{rs}^{(i)} b_{rs}^{(i)} = \lambda \sum_{r=1}^k \sum_{s=1}^\ell \left(\sum_{i=0}^{n-1} a_{rs}^{(i)} b_{rs}^{(i)} \right) = 0, \end{aligned}$$

which proves the result. \square

4. Polyadic Codes over the Ring \mathcal{R}

We now define polyadic codes of length n over the ring \mathcal{R} in terms of their idempotent generators with the assumption that the conditions on n and q for existence of polyadic codes over the field \mathbb{F}_q are satisfied. Let $\eta_{ij}, 1 \leq i \leq k, 1 \leq j \leq \ell$ be idempotents as defined in (3) and (4). Let the set of ordered suffixes $\{ij, 1 \leq i \leq k, 1 \leq j \leq \ell\}$ be divided into m disjoint subsets

$$\{ij, 1 \leq i \leq k, 1 \leq j \leq \ell\} = A_1 \cup A_2 \cup \dots \cup A_m \tag{7}$$

with the assumption that each of the sets A_i is non-empty, if $k\ell \geq m$. In that case let $|A_i| = r_i, 1 \leq r_i \leq k\ell - m + 1$.

If $k\ell < m$, we assume that in the partition (7), $k\ell$ sets are non-empty, each containing exactly one element and the remaining $m - k\ell$ sets are empty. Therefore $|A_i| = r_i = 1$, if A_i is non empty and $|A_i| = r_i = 0$, if A_i is empty.

Therefore

$$k\ell = r_1 + r_2 + \dots + r_m.$$

Define

$$\begin{aligned} \theta_{r_1} &= \sum_{ij \in A_1} \eta_{ij}, \\ \theta_{r_2} &= \sum_{ij \in A_2} \eta_{ij}, \\ &\vdots \\ \theta_{r_m} &= \sum_{ij \in A_m} \eta_{ij}, \end{aligned}$$

with the convention that empty sum is regarded as zero.

Using Lemma 2, we find that

$$\theta_{r_1} + \theta_{r_2} + \dots + \theta_{r_m} = 1, \tag{8}$$

and that $\theta_{r_i}, 1 \leq i \leq m$ are mutually orthogonal idempotents in the ring \mathcal{R} , i.e.,

$$\theta_{r_i}^2 = \theta_{r_i} \text{ for all } i, \theta_{r_i} \theta_{r_j} = 0, \text{ for all } i \neq j. \tag{9}$$

For $i = 1, 2, \dots, m$, let e_i, e'_i, d_i, d'_i be the idempotent generators of polyadic codes over \mathbb{F}_q as defined in Section 2.2.

For each tuple (r_1, r_2, \dots, r_m) , let

$$\begin{aligned}
 F_1 &= F_1^{(\eta_1, \eta_2, \dots, \eta_m)} = \theta_{\eta_1} d_1 + \theta_{\eta_2} d_2 + \dots + \theta_{\eta_m} d_m \\
 F_2 &= F_2^{(\eta_1, \eta_2, \dots, \eta_m)} = \mu_a(F_1) = \theta_{\eta_1} d_m + \theta_{\eta_2} d_1 + \dots + \theta_{\eta_m} d_{m-1} \\
 &\vdots \\
 F_m &= F_m^{(\eta_1, \eta_2, \dots, \eta_m)} = \mu_a(F_{m-1}) = \theta_{\eta_1} d_2 + \theta_{\eta_2} d_3 + \dots + \theta_{\eta_m} d_1 \\
 F'_1 &= F'_1{}^{(\eta_1, \eta_2, \dots, \eta_m)} = \theta_{\eta_1} d'_1 + \theta_{\eta_2} d'_2 + \dots + \theta_{\eta_m} d'_m \\
 F'_2 &= F'_2{}^{(\eta_1, \eta_2, \dots, \eta_m)} = \mu_a(F'_1) = \theta_{\eta_1} d'_m + \theta_{\eta_2} d'_1 + \dots + \theta_{\eta_m} d'_{m-1} \\
 &\vdots \\
 F'_m &= F'_m{}^{(\eta_1, \eta_2, \dots, \eta_m)} = \mu_a(F'_{m-1}) = \theta_{\eta_1} d'_2 + \theta_{\eta_2} d'_3 + \dots + \theta_{\eta_m} d'_1
 \end{aligned} \tag{10}$$

be odd-like idempotents in the ring $\mathcal{R}[x]/\langle x^n - 1 \rangle$. Similarly let

$$\begin{aligned}
 E_1 &= E_1^{(\eta_1, \eta_2, \dots, \eta_m)} = \theta_{\eta_1} e_1 + \theta_{\eta_2} e_2 + \dots + \theta_{\eta_m} e_m \\
 E_2 &= \mu_a(E_1), E_3 = \mu_a(E_2), \dots, E_m = \mu_a(E_{m-1}) \\
 E'_1 &= E'_1{}^{(\eta_1, \eta_2, \dots, \eta_m)} = \theta_{\eta_1} e'_1 + \theta_{\eta_2} e'_2 + \dots + \theta_{\eta_m} e'_m \\
 E'_2 &= \mu_a(E'_1), E'_3 = \mu_a(E'_2), \dots, E'_m = \mu_a(E'_{m-1})
 \end{aligned} \tag{11}$$

be even-like idempotents in the ring $\mathcal{R}[x]/\langle x^n - 1 \rangle$.

For each tuple (r_1, r_2, \dots, r_m) , and for each $i, 1 \leq i \leq m$, let $T_i^{(\eta_1, \eta_2, \dots, \eta_m)}, T'_i{}^{(\eta_1, \eta_2, \dots, \eta_m)}$ denote the odd-like polyadic codes and $P_i^{(\eta_1, \eta_2, \dots, \eta_m)}, P'_i{}^{(\eta_1, \eta_2, \dots, \eta_m)}$ denote the even-like polyadic codes over \mathcal{R} generated by the corresponding idempotents, *i.e.*

$$\begin{aligned}
 T_i^{(\eta_1, \eta_2, \dots, \eta_m)} &= \langle F_i^{(\eta_1, \eta_2, \dots, \eta_m)} \rangle, T'_i{}^{(\eta_1, \eta_2, \dots, \eta_m)} = \langle F'_i{}^{(\eta_1, \eta_2, \dots, \eta_m)} \rangle, \\
 P_i^{(\eta_1, \eta_2, \dots, \eta_m)} &= \langle E_i^{(\eta_1, \eta_2, \dots, \eta_m)} \rangle, P'_i{}^{(\eta_1, \eta_2, \dots, \eta_m)} = \langle E'_i{}^{(\eta_1, \eta_2, \dots, \eta_m)} \rangle.
 \end{aligned} \tag{12}$$

Clearly for any tuple (r_1, r_2, \dots, r_m) , $T_1^{(\eta_1, \eta_2, \dots, \eta_m)}, T_2^{(\eta_1, \eta_2, \dots, \eta_m)}, \dots, T_m^{(\eta_1, \eta_2, \dots, \eta_m)}$ are equivalent; $T'_1{}^{(\eta_1, \eta_2, \dots, \eta_m)}, T'_2{}^{(\eta_1, \eta_2, \dots, \eta_m)}, \dots, T'_m{}^{(\eta_1, \eta_2, \dots, \eta_m)}$ are equivalent; $P_1^{(\eta_1, \eta_2, \dots, \eta_m)}, P_2^{(\eta_1, \eta_2, \dots, \eta_m)}, \dots, P_m^{(\eta_1, \eta_2, \dots, \eta_m)}$ are equivalent and $P'_1{}^{(\eta_1, \eta_2, \dots, \eta_m)}, P'_2{}^{(\eta_1, \eta_2, \dots, \eta_m)}, \dots, P'_m{}^{(\eta_1, \eta_2, \dots, \eta_m)}$ are equivalent.

Next we compute the number of inequivalent odd-like and even-like polyadic codes over the ring \mathcal{R} .

Theorem 4: If $k\ell \geq m$, then there are

$$\frac{2}{m} \sum_{r_{m-1}=1}^{k\ell - (\eta_1 + \eta_2 + \dots + \eta_{m-2}) - 1} \dots \sum_{r_2=1}^{k\ell - \eta_1 - (m-2)} \sum_{\eta_1=1}^{k\ell - (m-1)} \binom{k\ell}{r_1} \binom{k\ell - r_1}{r_2} \dots \binom{k\ell - (r_1 + r_2 + \dots + r_{m-2})}{r_{m-1}}$$

inequivalent odd-like polyadic codes and the same number of inequivalent even-like polyadic codes over the ring \mathcal{R} .

If $k\ell < m$, then there are

$$\frac{2}{m} (k\ell)! \binom{m}{k\ell}$$

inequivalent odd-like polyadic codes and the same number of inequivalent even-like polyadic codes over the ring \mathcal{R} .

Proof: Let first $k\ell \geq m$, out of $k\ell$ idempotents $\eta_{ij}, 1 \leq i \leq k, 1 \leq j \leq \ell$, θ_{η_1}

can be chosen in $\binom{k\ell}{r_1}$ ways. Out of remaining $(k\ell - r_1)$ idempotents θ_{r_2} can be chosen in $\binom{k\ell - r_1}{r_2}$ ways, continuing like this $\theta_{r_{m-1}}$ can be chosen in $\binom{k\ell - (r_1 + r_2 + \dots + r_{m-2})}{r_{m-1}}$ ways and θ_{r_m} will be fixed. As each $\theta_{r_i}, 1 \leq i \leq m$ must have at least one η_{ij} , the number of choices of idempotents

$$\theta_{r_1} d_1 + \theta_{r_2} d_2 + \dots + \theta_{r_m} d_m \text{ is } \sum_{r_{m-1}=1}^{k\ell - (r_1 + r_2 + \dots + r_{m-2}) - 1} \dots \sum_{r_2=1}^{k\ell - r_1 - (m-2)} \sum_{r_1=1}^{k\ell - (m-1)} \binom{k\ell}{r_1} \binom{k\ell - r_1}{r_2} \dots \binom{k\ell - (r_1 + r_2 + \dots + r_{m-2})}{r_{m-1}}.$$

Since $\mu_a(F_1) = F_2, \mu_a(F_2) = F_3, \dots, \mu_a(F_m) = F_1$, and $F_i'^{(\eta_1, \eta_2, \dots, \eta_m)}$'s contribute equal number of inequivalent odd-like idempotents, we get the desired number.

Let now $k\ell < m$. Firstly the $k\ell$ non-empty sets A_i in the partition (7) can be chosen in $\binom{m}{k\ell}$ ways. Out of $k\ell$ idempotents $\eta_{ij}, 1 \leq i \leq k, 1 \leq j \leq \ell$, first non-zero θ_{r_i} can be chosen in $k\ell$ ways, next non-zero θ_{r_j} can be chosen in $k\ell - 1$ ways, ..., so the number of choices of $F_1 = \theta_{r_1} d_1 + \theta_{r_2} d_2 + \dots + \theta_{r_m} d_m$ is $(k\ell)! \binom{m}{k\ell}$. Since $\mu_a(F_1) = F_2, \mu_a(F_2) = F_3, \dots, \mu_a(F_m) = F_1$, and F_i' 's contribute equal number of inequivalent odd-like idempotents, we get the required number. \square

We drop the superscript (r_1, r_2, \dots, r_m) , when there is no confusion with the idempotents or the corresponding polyadic codes.

Theorem 5: For any subset $\{t_1, t_2, \dots, t_r\} \subseteq A = \{1, 2, \dots, m\}$, where $2 \leq r \leq m$, the following assertions hold for polyadic codes over \mathcal{R} .

- 1) $T_1 \cap T_2 \cap \dots \cap T_m = \langle \bar{j}(x) \rangle$, the repetition code over \mathcal{R}
- 2) $T_1 + T_2 + \dots + T_m = T_{t_1} + T_{t_2} + \dots + T_{t_r}$,
- 3) $P_1 \cap P_2 \cap \dots \cap P_m = P_{r_1} \cap P_{r_2} \dots \cap P_{r_r}$,
- 4) $P_1 + P_2 + \dots + P_m = \langle 1 - \bar{j}(x) \rangle$, the even weight code over \mathcal{R} ,
- 5) $P_i \cap \langle \bar{j}(x) \rangle = \{0\}, T_i \cap \langle \bar{j}(x) \rangle = \langle \bar{j}(x) \rangle$,
- 6) $P_i + T_i = \mathcal{R}[x] / \langle x^n - 1 \rangle, P_i \cap T_i = \{0\}$.

Proof: From the definitions and relations (8)-(11), we find that the sums of products of terms from F_1, F_2, \dots, F_m taken one at a time, taken two at a time and so on is equal to the sums of products of terms from d_1, d_2, \dots, d_m taken one at a time, taken two at a time and so on, *i.e.*

$$\begin{aligned} F_1 + F_2 + \dots + F_m &= d_1 + d_2 + \dots + d_m, \\ \sum_{\substack{i,j \\ i < j}} F_i(x) F_j(x) &= \sum_{\substack{i,j \\ i < j}} d_i(x) d_j(x), \\ \sum_{\substack{i,j,k \\ i < j < k}} F_i(x) F_j(x) F_k(x) &= \sum_{\substack{i,j,k \\ i < j < k}} d_i(x) d_j(x) d_k(x), \\ &\vdots \\ F_1 F_2 \dots F_m &= d_1 d_2 \dots d_m \end{aligned} \tag{13}$$

Therefore by Lemma 1 and relations (13), we find that

$$T_1 \cap T_2 \cap \dots \cap T_m = \langle F_1 F_2 \dots F_m \rangle = \langle d_1 d_2 \dots d_m \rangle,$$

$$T_1 + T_2 + \dots + T_m = \left\langle \sum_{i=1}^m d_i - \sum_{i < j} d_i d_j + \sum_{i < j < k} d_i d_j d_k - \dots (-1)^{m-1} \prod_{i=1}^m d_i \right\rangle,$$

$$T_{t_1} + T_{t_2} + \dots + T_{t_r} = \left\langle \sum_{i=1}^r d_{t_i} - \sum_{t_i < t_j} d_{t_i} d_{t_j} + \sum_{t_i < t_j < t_k} d_{t_i} d_{t_j} d_{t_k} - \dots (-1)^{r-1} \prod_{i=1}^r d_{t_i} \right\rangle.$$

By Proposition 1 (8) and (9), we get (1) and (2).

To prove (3), from Proposition 1 (6) we see that

$$\begin{aligned} E_1 E_2 &= \theta_{t_1} (e_1 e_m) + \theta_{t_2} (e_2 e_1) + \dots + \theta_{t_m} (e_m e_{m-1}) \\ &= \theta_{t_1} (e_1 e_2 \dots e_m) + \theta_{t_2} (e_1 e_2 \dots e_m) + \dots + \theta_{t_m} (e_1 e_2 \dots e_m) \\ &= e_1 e_2 \dots e_m. \end{aligned}$$

Similarly $E_{t_1} E_{t_2} \dots E_{t_r} = e_1 e_2 \dots e_m$ for any tuple (t_1, t_2, \dots, t_r) . Hence $E_{t_1} E_{t_2} \dots E_{t_r} = E_1 E_2 \dots E_m$, so we get (3) by Lemma 1.

Again as $E_1 + E_2 + \dots + E_m = e_1 + e_2 + \dots + e_m$ and for any tuple (t_1, t_2, \dots, t_r) , $E_{t_1} E_{t_2} \dots E_{t_r} = e_1 e_2 \dots e_m$, we see that

$$\begin{aligned} P_1 + P_2 + \dots + P_m &= \left\langle \sum_{i=1}^m E_i - \sum_{i < j} E_i E_j + \sum_{i < j < k} E_i E_j E_k - \dots (-1)^{m-1} \prod_{i=1}^m E_i \right\rangle \\ &= \left\langle \sum_{i=1}^m e_i - \binom{m}{2} e_1 e_2 \dots e_m + \binom{m}{3} e_1 e_2 \dots e_m - \dots (-1)^{m-1} \binom{m}{m} e_1 e_2 \dots e_m \right\rangle \\ &= \langle e_1 + e_2 + \dots + e_m - (m-1) e_1 e_2 \dots e_m \rangle. \end{aligned}$$

Now (4) follows from Proposition 1 (7).

Since $e_j(\bar{j}(x)) = 0$ for all $1 \leq j \leq m$ by Proposition 2 (15), we get $E_i(\bar{j}(x)) = 0$ and so $P_i \cap \langle \bar{j}(x) \rangle = \{0\}$. As $d_i = 1 - e_i$, we find that $F_i = 1 - E_i$ and so $F_i(\bar{j}(x)) = \bar{j}(x) - E_i(\bar{j}(x)) = \bar{j}(x)$. Therefore $T_i \cap \langle \bar{j}(x) \rangle = \langle \bar{j}(x) \rangle$. This proves (5).

We prove (6) for $i = 1$. Others are similar. Note that

$$\begin{aligned} E_1 F_1 &= \theta_{t_1} (e_1 d_1) + \theta_{t_2} (e_2 d_2) + \dots + \theta_{t_m} (e_m d_m) = 0 \text{ and} \\ E_1 + F_1 &= \theta_{t_1} (e_1 + d_1) + \theta_{t_2} (e_2 + d_2) + \dots + \theta_{t_m} (e_m + d_m) = 1, \text{ by Proposition 1 (10).} \end{aligned}$$

Therefore $P_1 \cap T_1 = \langle E_1 F_1 \rangle = \{0\}$ and $P_1 + T_1 = \langle E_1 + F_1 - E_1 F_1 \rangle = \langle 1 \rangle$. \square

Similarly we have.

Theorem 6: For any subset $\{t_1, t_2, \dots, t_r\} \subseteq A$, where $2 \leq r \leq m$, the following assertions hold for polyadic codes over \mathcal{R} .

- 1) $T'_1 \cap T'_2 \cap \dots \cap T'_m = T'_{t_1} \cap T'_{t_2} \cap \dots \cap T'_{t_r}$,
- 2) $T'_1 + T'_2 + \dots + T'_m = \langle 1 \rangle = \mathcal{R}[x] / \langle x^n - 1 \rangle$,
- 3) $P'_1 \cap P'_2 \cap \dots \cap P'_m = \{0\}$,
- 4) $P'_1 + P'_2 + \dots + P'_m = P'_{t_1} + P'_{t_2} + \dots + P'_{t_r}$,
- 5) $P'_i \cap \langle \bar{j}(x) \rangle = \{0\}$, $T'_i \cap \langle \bar{j}(x) \rangle = \langle \bar{j}(x) \rangle$,
- 6) $P'_i + T'_i = \mathcal{R}[x] / \langle x^n - 1 \rangle$, $P'_i \cap T'_i = \{0\}$,

- 7) $P_i + \langle \bar{j}(x) \rangle = T_i', P_i + \langle \bar{j}(x) \rangle = T_i,$
- 8) $P_i + P_i' = \langle 1 - \bar{j}(x) \rangle, P_i \cap P_i' = \{0\}$ and,
- 9) $T_i + T_i' = \mathcal{R}[x] / \langle x^n - 1 \rangle, T_i \cap T_i' = \langle \bar{j}(x) \rangle.$

Proof: The proof of statements (1) to (6) is similar to that of (1) to (6) of Theorem 5. To prove (7) we note that

$$\begin{aligned} E_1 + \bar{j}(x) - E_1(\bar{j}(x)) &= E_1 + \bar{j}(x) \\ &= \theta_{r_1} e_1 + \theta_{r_2} e_2 + \dots + \theta_{r_m} e_m + \bar{j}(x)(\theta_{r_1} + \theta_{r_2} + \dots + \theta_{r_m}) \\ &= \theta_{r_1}(e_1 + \bar{j}(x)) + \theta_{r_2}(e_2 + \bar{j}(x)) + \dots + \theta_{r_m}(e_m + \bar{j}(x)) \\ &= \theta_{r_1} d_1' + \theta_{r_2} d_2' + \dots + \theta_{r_m} d_m' = F_1', \text{ by Proposition 2 (15).} \end{aligned}$$

Hence $P_i + \langle \bar{j}(x) \rangle = T_i'.$ Similarly others. Statements (8) and (9) follow from Proposition 2 (16). \square

Theorem 7: Let $P_i, P_i',$ for $i = 1, 2, \dots, m,$ be two pairs of even-like polyadic codes over the ring with T_i, T_i' the associated pairs of odd-like polyadic codes. Then

$$P_i^\perp = \mu_{-1}(T_i) \text{ and } P_i'^\perp = \mu_{-1}(T_i').$$

Further if $\mu_{-1}(e_i) = e_i$ for $i = 1, 2, \dots, m,$ then

$$P_i^\perp = T_i, P_i'^\perp = T_i' \text{ and } P_i, P_i', T_i, T_i' \text{ are LCD codes over } \mathcal{R}.$$

Proof: By Proposition 1 (10), $e_i + d_i = 1.$ So $\mu_{-1}(e_i) + \mu_{-1}(d_i) = \mu_{-1}(1) = 1.$ Therefore

$$\begin{aligned} 1 - \mu_{-1}(E_1) &= \theta_{r_1} + \theta_{r_2} + \dots + \theta_{r_m} - \mu_{-1}(\theta_{r_1} e_1 + \theta_{r_2} e_2 + \dots + \theta_{r_m} e_m) \\ &= \theta_{r_1}(1 - \mu_{-1}(e_1)) + \theta_{r_2}(1 - \mu_{-1}(e_2)) + \dots + \theta_{r_m}(1 - \mu_{-1}(e_m)) \\ &= \theta_{r_1} \mu_{-1}(d_1) + \theta_{r_2} \mu_{-1}(d_2) + \dots + \theta_{r_m} \mu_{-1}(d_m) \\ &= \mu_{-1}(\theta_{r_1} d_1 + \theta_{r_2} d_2 + \dots + \theta_{r_m} d_m) = \mu_{-1}(F_1). \end{aligned}$$

Hence $P_1^\perp = \langle 1 - \mu_{-1}(E_1) \rangle = \langle \mu_{-1}(F_1) \rangle = \mu_{-1}(\langle F_1 \rangle) = \mu_{-1}(T_1).$ Similarly, we get the others.

Further if $\mu_{-1}(e_i) = e_i$ for $i = 1, 2, \dots, m,$ then by Theorem 5 (6) and Theorem 6 (6),

$$P_i \cap P_i^\perp = P_i \cap T_i = \{0\}, P_i' \cap P_i'^\perp = P_i' \cap T_i' = \{0\}$$

proving thereby that P_i and P_i' are LCD codes over $\mathcal{R}.$ Similarly one can check that T_i and T_i' are also LCD codes over $\mathcal{R}.$ \square

Theorem 8: If S'_∞ is empty, then we have the following additional results:

- 1) $|P_i| = q^{\frac{(\ell k)(n-1)}{m}}, |T_i| = q^{\frac{(\ell k)(n+m-1)}{m}}.$
- 2) $|P_i| = q^{\frac{(\ell k)(n-1)(m-1)}{m}}, |T_i| = q^{\frac{(\ell k)(mn-n+1)}{m}}.$

Proof: Here since $e_{r_1} e_{r_2} \dots e_{r_r} = 0,$ by Proposition 3, we have $E_{r_1} E_{r_2} \dots E_{r_r} = 0$ for ant tuple $(t_1, t_2, \dots, t_r).$ Therefore for any $s, 1 \leq s \leq m-1,$

$$\begin{aligned} P_1 + P_2 + \dots + P_s &= \langle E_1 + E_2 + \dots + E_s \rangle \text{ and} \\ (P_1 + P_2 + \dots + P_s) \cap P_{s+1} &= \langle (E_1 + E_2 + \dots + E_s) E_{s+1} \rangle = \{0\}. \end{aligned}$$

Hence by proposition 5 (4),

$$\begin{aligned} \left| \langle 1 - \bar{j}(x) \rangle \right| &= |P_1 + P_2 + \dots + P_m| = \frac{|P_1 + P_2 + \dots + P_{m-1}| |P_m|}{|(P_1 + P_2 + \dots + P_{m-1}) \cap P_m|} \\ &= |P_1 + P_2 + \dots + P_{m-1}| |P_m| = \frac{|P_1 + P_2 + \dots + P_{m-2}| |P_{m-1}|}{|(P_1 + P_2 + \dots + P_{m-2}) \cap P_{m-1}|} |P_m| \\ &= |P_1 + P_2 + \dots + P_{m-2}| |P_{m-1}| |P_m| = \dots = \frac{|P_1|}{q^{\binom{\ell k}{m}}} |P_2| \dots |P_{m-1}| |P_m| = |P_1|^m. \end{aligned}$$

As $\left| \langle 1 - \bar{j}(x) \rangle \right| = (q^{\ell k})^{\binom{n-1}{m}}$, we get that $|P_1| = q^{\frac{\binom{\ell k}{m}(n-1)}{m}}$. Since from Theorems 5 and 6, we have $P_i + \langle \bar{j}(x) \rangle = T_i'$ and $P_i \cap \langle \bar{j}(x) \rangle = \{0\}$,

$$|T_i'| = |P_i| \left| \langle \bar{j}(x) \rangle \right| = q^{\frac{\binom{\ell k}{m}(n-1)}{m}} q^{\ell k} = q^{\frac{\binom{\ell k}{m}(m+n-1)}{m}}.$$

Again from Theorem 6 (8), we see that $|P_i| |P_i'| = \left| \langle 1 - \bar{j}(x) \rangle \right| = q^{\binom{\ell k}{m}(n-1)}$, which gives $|P_i'| = q^{\frac{\binom{\ell k}{m}(n-1)(m-1)}{m}}$. Finally $P_i' \oplus \langle \bar{j}(x) \rangle = T_i$ gives

$$|T_i| = |P_i'| \left| \langle \bar{j}(x) \rangle \right| = q^{\frac{\binom{\ell k}{m}(n-1)(m-1)}{m}} q^{\ell k} = q^{\frac{\binom{\ell k}{m}(m+n-1)}{m}}.$$

4.1. Extensions of Polyadic Codes over the Ring \mathcal{R}

When S'_∞ is empty, we consider extended polyadic codes over the ring \mathcal{R} which give us some additional results.

Consider the equation

$$1 + \gamma^2 n = 0. \tag{14}$$

This equation has a solution γ in \mathbb{F}_q if and only if n and -1 are both squares or both non squares in \mathbb{F}_q (see [17], Chapter 6).

For a linear code C of length n over \mathcal{R} , \bar{C} , the extension of C is defined as

$$\bar{C} = \left\{ (c_0, c_1, \dots, c_{n-1}, c_\infty) : c_\infty = \gamma \sum_{j=0}^{n-1} c_j, (c_0, c_1, \dots, c_{n-1}) \in C \right\}.$$

Theorem 9: Let S'_∞ be empty. Suppose there exists a γ in \mathbb{F}_q satisfying Equation (13). If the splitting of \mathbb{Z}_n in (1) is given by the multiplier μ_{-1} , then the extended odd-like polyadic codes satisfy $\overline{T'_{i+1}} - \overline{T_i}^\perp$.

Proof: Here, by Theorem 7, $P_i'^\perp = \mu_{-1}(T_i') = T_{i+1}'$. As $T_i = P_i' + \langle \bar{j}(x) \rangle$ and $T_{i+1}' = P_{i+1}' + \langle \bar{j}(x) \rangle$, by Theorem 6 (7), let $\overline{T_i}$ and $\overline{T'_{i+1}}$ be the extended polyadic code over \mathcal{R} generated by

$$\overline{G_i} = \begin{pmatrix} 0 & 1 & 2 & \dots & n-1 & \infty \\ & & & & & 0 \\ & & & & & 0 \\ \vdots & & & & & \vdots \\ 1 & 1 & 1 & \dots & 1 & -n\gamma \end{pmatrix}$$

and

$$\overline{G'_{i+1}} = \begin{pmatrix} 0 & 1 & 2 & \dots & n-1 & \infty \\ & & & & & 0 \\ & & & & & 0 \\ \vdots & & & & & \vdots \\ 1 & 1 & 1 & \dots & 1 & -n\gamma \end{pmatrix}$$

where G'_i is a generator matrix for the even-like polyadic code P'_i and G_{i+1} is a generator matrix for the even-like polyadic code P_{i+1} . The row above the matrix shows the column labeling by $\mathbb{Z}_n \cup \infty$. Since the all one vector belongs to T'_{i+1} and its dual T'^{\perp}_{i+1} is equal to P'_i , the last row of $\overline{G'_{i+1}}$ is orthogonal to all rows of G'_i . The last row is orthogonal to itself also as $n + \gamma^2 n^2 = 0$ in \mathbb{F}_q . Therefore all rows of $\overline{G'_{i+1}}$ are orthogonal to all the rows of $\overline{G_i}$. Now the result follows from the fact that $\left| \overline{T'_{i+1}} \right| = \left| \overline{T_i}^{\perp} \right|$, as can be verified from Theorem 8. \square

Similarly, we have

Theorem 10: Let S'_∞ be empty. Suppose there exists a γ in \mathbb{F}_q satisfying Equation (14). If $\mu_{-1}(e_i) = e_i$ so that the splitting of \mathbb{Z}_n in (2) is not given by the multiplier μ_{-1} , then the extended odd-like polyadic codes satisfy $\overline{T_i}^{\perp} = \overline{T'_i}$.

Corollary 1: If S'_∞ is empty, $m = 2$ then the following assertions hold for duadic codes over \mathcal{R} .

1) If $\mu_{-1}(e_1) = e_2, \mu_{-1}(e_2) = e_1$, then $P_i^{\perp} = T'_i, P_i'^{\perp} = T$; P_i are self orthogonal and $\overline{T_i}$ are self dual.

2) If $\mu_{-1}(e_1) = e_1, \mu_{-1}(e_2) = e_2$ then $\overline{T_i}$ are isodual.

Proof: Here, by definition $\mathbb{C}'_1 = \mathbb{C}_2$ and $\mathbb{D}'_1 = \mathbb{D}_2$, therefore $E'_1 = E_2$, $F'_1 = F_2$, $T'_1 = T_2$, $T'_2 = T_1$, $P'_1 = P_2$ and $P'_2 = P_1$.

If $\mu_{-1}(e_1) = e_2, \mu_{-1}(e_2) = e_1$, i.e., when the splitting is given by μ_{-1} , we have by Theorem 7, $P_i^{\perp} = T_{i+1}$, subscript modulo m . Therefore

$P_1^{\perp} = T_2 = T'_1, P_2^{\perp} = T_1 = T'_2$. Using statement (7) of Theorem 6, we have $P_i \subseteq T'_i = P_i^{\perp}$. Therefore P_i is self-orthogonal. By Theorem 9, $\overline{T'_{i+1}} = \overline{T_i}^{\perp}$, therefore $\overline{T_i}^{\perp} = \overline{T_i}$.

If $\mu_{-1}(e_i) = e_i$, By Theorem 10, $\overline{T_i}^{\perp} = \overline{T'_i}$, therefore $\overline{T_1}^{\perp} = \overline{T_2}$ and $\overline{T_2}^{\perp} = \overline{T_1}$.

4.2. Griesmer Type Bound for Polyadic Codes over \mathcal{R}

Kuruz *et al.* [11] gave some examples of m -adic residue codes over $\mathbb{F}_q[u]/\langle u^2 - u \rangle$ whose parameters attain Griesmer type bound. In the next theorem, we prove that the Griesmer type bound for polyadic codes over the ring \mathcal{R} is same as the Griesmer bound for the corresponding polyadic codes over the field \mathbb{F}_q .

Theorem 11: The parameters of polyadic codes over \mathcal{R} are same as parameters of the corresponding polyadic codes over \mathbb{F}_q . Hence Griesmer type bound for polyadic codes over the ring \mathcal{R} is same as the Griesmer bound for the corresponding polyadic codes over the field \mathbb{F}_q .

Proof: Let \mathcal{C} be a polyadic code of length n over $\mathcal{R} = \bigoplus \eta_{ij} \mathbb{F}_q$. Then \mathcal{C} is equal to T_i or T'_i or P_i or P'_i for $1 \leq i \leq m$. By definition,

$T_1 = \theta_1 \mathbb{D}_1 \oplus \theta_2 \mathbb{D}_2 \oplus \dots \oplus \theta_m \mathbb{D}_m$, where $\mathbb{D}_1, \mathbb{D}_2, \dots, \mathbb{D}_m$ are all odd-like polyadic codes over \mathbb{F}_q and are equivalent. Therefore by Theorem 1,

$$k(T_1) = \max_{i=1}^m \left\{ k(\theta_i \mathbb{D}_i) \right\} = \text{dimension of polyadic code } \mathbb{D}_i = k(\mathbb{D}_i)$$

$$d(T_1) = \min_{i=1}^m \left\{ d(\theta_i \mathbb{D}_i) \right\} = \text{minimum distance of polyadic code } \mathbb{D}_i = d(\mathbb{D}_i).$$

Further $T_i = \mu_{\alpha^{i-1}}(T_1)$. Here Jacobson radical, $J(\eta_{ij} \mathbb{F}_q) = \{0\}$, so

$|\eta_{ij}\mathbb{F}_q/J(\eta_{ij}\mathbb{F}_q)|=q$ for every i and j . Hence the Griesmer type bound for odd-like polyadic codes T_i over the ring \mathcal{R} becomes

$$n \geq \sum_{i=0}^{k(\mathbb{D}_i)-1} \left\lceil \frac{d(\mathbb{D}_i)}{q^i} \right\rceil,$$

which is same as Griesmer bound for polyadic code \mathbb{D}_i over \mathbb{F}_q .

Similar result holds for T'_i, P_i and P'_i .

Example 1: Let $q=3, n=13, m=4, f(u)=u^3-u$ and $g(v)=v^2-1$. Take $E_1=(\eta_{11}+\eta_{12})e_1+(\eta_{21}+\eta_{22})e_2+(\eta_{31})e_3+(\eta_{32})e_4$. Here

$P_1=(\eta_{11}+\eta_{12})\mathbb{C}_1 \oplus (\eta_{21}+\eta_{22})\mathbb{C}_2 \oplus (\eta_{31})\mathbb{C}_3 \oplus (\eta_{32})\mathbb{C}_4$ has parameters $[13,3,9]$. It attains the Griesmer type bound over the ring

$\mathbb{F}_3[u,v]/\langle u^3-u, v^2-1, uv-vu \rangle$. Therefore $P_i, i=1,2,3,4$ are optimal. The code $T_1=(\eta_{11}+\eta_{12})\mathbb{D}_1 \oplus (\eta_{21}+\eta_{22})\mathbb{D}_2 \oplus (\eta_{31})\mathbb{D}_3 \oplus (\eta_{32})\mathbb{D}_4$ has parameters $[13,10,3]$. It nearly attains Griesmer type bound over the ring

$\mathbb{F}_3[u,v]/\langle u^3-u, v^2-1, uv-vu \rangle$.

Example 2: Let $q=5, n=11, m=2, f(u)=u^3-u$ and $g(v)=v^2-1$. Here $P_1=(\eta_{11}+\eta_{12}+\eta_{21}+\eta_{22})\mathbb{C}_1 \oplus (\eta_{31}+\eta_{32})\mathbb{C}_2$ has parameters $[11,5,6]$, so it attains the Griesmer type bound over the ring $\mathbb{F}_5[u,v]/\langle u^3-u, v^2-1, uv-vu \rangle$. Therefore P_1 and P_2 are optimal.

Remark: Using the above theory, one can construct some other cyclic codes over the ring \mathcal{R} (which are not polyadic according to our definition) generated by idempotents of the type

$$\theta_{r_1} \left(\sum_{i \in I_1} e_i \right) + \theta_{r_2} \left(\sum_{i \in I_2} e_i \right) + \dots + \theta_{r_m} \left(\sum_{i \in I_m} e_i \right)$$

where I_1, I_2, \dots, I_m are subsets of $\{1, 2, \dots, m\}$, and which may attain the Griesmer type bound.

For example, take $q=11, n=5, m=4, f(u)=(u^2-1)(u-2), g(v)=v^2-v$ and $E=(\eta_{11}+\eta_{12}+\eta_{21})(e_1+e_2+e_3)+(\eta_{22}+\eta_{31}+\eta_{32})(e_1+e_2)$. Let C be a cyclic code over ring \mathcal{R} generated by the idempotent E , then C has parameters $[5,3,3]$ and it attains the Griesmer type bound.

As an another example, take $q=7, n=19, m=6, f(u)=u^4-u$ and $g(v)=v^2-v$ and

$E_1=(\eta_{11}+\eta_{12}+\eta_{21})(e_1+e_2)+(\eta_{22}+\eta_{31}+\eta_{32})(e_2+e_3)+(\eta_{41}+\eta_{42})(e_3+e_5)$. The cyclic code C generated by the idempotent E_1 over ring \mathcal{R} has parameters $[19,12,6]$ and it nearly attains the Griesmer type bound.

4.3. Gray Images of Polyadic Codes over \mathcal{R}

Theorem 12: Let the matrix V taken in the definition of the Gray map Φ satisfy $VV^T = \lambda I_m, \lambda \in \mathbb{F}_q^*$. For all possible choices of (r_1, \dots, r_m) , the Gray images of even-like polyadic codes $P_i^{(r_1, \dots, r_m)}, P'_i^{(r_1, \dots, r_m)}$ and Gray images of extensions of odd-like polyadic codes $T_i^{(r_1, \dots, r_m)}, T'_i^{(r_1, \dots, r_m)}$, for $i=1, 2, \dots, m$, have the following properties

- 1) If $\mu_{-1}(e_i) = e_i$, i.e., if $\mu_{-1}(S_i) = S_i$, then $\Phi(P_i)$, $\Phi(P'_i)$, $\Phi(T_i)$ and $\Phi(T'_i)$ are linear complementary dual(LCD) codes of length $k\ell n$ over \mathbb{F}_q .
- 2) If S'_∞ is empty and $\mu_{-1}(S_i) = S_i$, then $\Phi(\overline{T'_i}) = (\Phi(\overline{T_i}))^\perp$.
- 3) If S'_∞ is empty and $\mu_{-1}(S_i) = S_{i+1}$, i.e., if the splitting in (1) is given by μ_{-1} then $\Phi(\overline{T'_{i+1}}) = (\Phi(\overline{T_i}))^\perp$.

The theorem follows from Theorems 3, 7, 9 and 10.

Corollary 2: If S'_∞ is empty, $m = 2$, then the following assertions hold for duadic codes over \mathcal{R} .

- 1) If $\mu_{-1}(e_1) = e_2, \mu_{-1}(e_2) = e_1$, then $\Phi(P_i)$ are self orthogonal of length $k\ell n$ and $\Phi(\overline{T_i})$ are self-dual codes of length $k\ell(n+1)$ over \mathbb{F}_q .
- 2) If $\mu_{-1}(e_1) = e_1, \mu_{-1}(e_2) = e_2$, then $\Phi(\overline{T_i})$ are isodual codes of length $k\ell(n+1)$ over \mathbb{F}_q .

The following examples illustrate our theory. The minimum distances of all these codes have been computed by the Magma Computational Algebra System.

Example 3: Let $q = 13$, $n = 3$, $m = 2$, $f(u) = u^2 - u$, $g(v) = v^3 - v$, $\gamma = 2$ and

$$V = A = \begin{pmatrix} 2 & -2 & 1 & 2 & -2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 1 & -2 & 2 & 1 & -2 \\ 2 & -2 & 1 & -2 & 2 & -1 \\ 1 & 2 & 2 & -1 & -2 & -2 \\ 2 & 1 & -2 & -2 & -1 & 2 \end{pmatrix}$$

be a matrix over \mathbb{F}_{13} satisfying $VV^T = 5I$. Here $S'_\infty = \emptyset$, $e_1 = 3x^2 + x + 9$ and $e_2 = x^2 + 3x + 9$. Also $\mu_{-1}(e_1) = e_2$ and $\mu_{-1}(e_2) = e_1$. On taking

$\theta_1 = \eta_{11} + \eta_{12} + \eta_{13} + \eta_{21} + \eta_{22}$ and $\theta_2 = \eta_{23}$, we have

$E_1 = -x^2(uv^2 + uv - 3) + x(uv^2 + uv + 1) + 9$ and

$F_1 = -x^2(uv^2 + uv + 1) + x(uv^2 + uv - 3) + 5$. The Gray images of polyadic codes $P_1^{(5,1)}$ and $\overline{T_1}^{(5,1)}$ are self-orthogonal [18, 6, 6] and self-dual [24, 12, 4] codes over \mathbb{F}_{13} respectively.

Example 4: Let $q = 4$, $n = 17$, $m = 4$, $f(u) = u^2 - 1$, $g(v) = v^2 - v$ and

$$V = B = \begin{pmatrix} a & -a^2 & 1 & 1 \\ -1 & 1 & a & a^2 \\ a^2 & a & -1 & 1 \\ 1 & 1 & a^2 & -a \end{pmatrix}$$

be a matrix over \mathbb{F}_4 satisfying $VV^T = I$, where a is a primitive element of \mathbb{F}_4 . The Gray images of polyadic codes $P_1^{(1,2,1)}$, $T_1^{(1,2,1)}$, $P_1^{r(1,2,1)}$ and $T_1^{r(1,2,1)}$ with $\theta_1 = \eta_{11}, \theta_2 = \eta_{12} + \eta_{21}$ and $\theta_3 = \eta_{22}$ are respectively [68, 16, 28], [68, 52, 6], [68, 48, 8] and [68, 20, 17] LCD codes over \mathbb{F}_4 .

Some other examples of self-dual, self-orthogonal and LCD codes arising as Gray images of Polyadic codes over \mathcal{R} are given in **Table 1**. The matrices A and B in **Table 1** are as defined in Examples 3 and 4 respectively and the matrix C is taken as

$$C = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 2 & 1 & -1 \\ 1 & 1 & -2 & 2 \end{pmatrix}.$$

Table 1. Gray images of some polyadic codes.

q	n	m	$f(u)$	$g(v)$	V	γ	(r_1, \dots, r_m)	$\Phi(P_1)$	$\Phi(\overline{T}_1)$
4	13	2	$u^2 - u$	$v^2 - v$	B	1	(3, 1)	[52, 24, 12]	[56, 28, 12]
								LCD	isodual
5	11	2	$u^2 - 1$	$v^2 - 1$	H_4^\dagger	2	(3, 1)	[44, 20, 12]	[48, 24, 11]
								self-orthogonal	self-dual
7	9	2	$u^2 - 1$	$v^3 - v$	A	does not exist	(5, 1)	[54, 24, 6]	
								self-orthogonal	
7	3	2	$u^2 - u$	$v^3 - v$	A	3	(5, 1)	[18, 6, 6]	[24, 12, 4]
								self-orthogonal	self-dual
11	5	2	$u^2 - u$	$v^3 - v$	A	does not exist	(5, 1)	[30, 12, 8]	
								self-orthogonal	

q	n	m	$f(u)$	$g(v)$	V	(r_1, \dots, r_m)	$\Phi(P_1)$	$\Phi(T_1)$	$\Phi(P_1')$	$\Phi(T_1')$
3	13	4	$u^2 - 1$	$v^2 - 1$	H_4	(1, 1, 1, 1)	[52, 12, 24]	[52, 40, 5]	[52, 36, 8]	[52, 16, 13]
								$\sim \Phi(P_1)^\perp$	$\sim \Phi(P_1')^\perp$	
5	13	3	$u^2 - 1$	$v^2 - 1$	H_4	(2, 1, 1)	[52, 16, 16]	[52, 36, 7]	[52, 32, 8]	[52, 20, 13]
							LCD	LCD	LCD	LCD
7*	16	2	$u^2 - 1$	$v^2 - v$	H_4	(3, 1)	[64, 44, 4]	[64, 20, 12]	[64, 16, 12]	[64, 48, 2]
							LCD	LCD	LCD	LCD
11	5	4	$u^2 - u$	$v^2 - 1$	C	(1, 1, 1, 1)	[20, 4, 14]	[20, 16, 4]	[20, 12, 6]	[20, 8, 5]
								$\sim \Phi(P_1)^\perp$	$\sim \Phi(P_1')^\perp$	
13	17	4	$u^2 - 1$	$v^2 - v$	F	(1, 1, 1, 1)	[68, 16, 12]	[68, 52, 4]	[68, 48, 4]	[68, 20, 11]
							LCD	LCD	LCD	LCD
16	17	4	$u^2 - u$	$v^2 - v$	I	(1, 1, 1, 1)	[68, 16, 14]	[68, 48, 5]	[68, 52, 5]	[68, 20, 11]
							LCD	LCD	LCD	LCD
32	11	5	$u^2 - u$	$v^2 - v$	I	(1, 1, 1, 1, 0)	[44, 8, 10]	[44, 32, 4]	[44, 36, 3]	[44, 12, 9]
							LCD	LCD	LCD	LCD

*In this case, $S_\infty' = \{2, 4, 6, 8, 10, 12, 14\}$.

[†] H_4 is Hadamard matrix of order 4.

[‡] I is the Identity matrix.

5. Conclusion

In this paper, polyadic codes and their extensions over a finite commutative non-chain ring $\mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu \rangle$ are studied where $f(u)$ and $g(v)$ are two polynomials of degree k and ℓ respectively (k and ℓ are not both 1) which split into distinct linear factors over \mathbb{F}_q . A Gray map is defined from $\mathcal{R}^n \rightarrow \mathbb{F}_q^{k\ell n}$ which preserves duality. As a consequence, self-dual, isodual, self-orthogonal and complementary dual (LCD) codes over \mathbb{F}_q are constructed. Some examples are also given to illustrate our theory. It is shown that the Griesmer type bound for polyadic codes over the ring \mathcal{R} is same as the Griesmer bound for the corresponding polyadic codes over the field \mathbb{F}_q . Examples of some codes which are optimal with respect to Griesmer type bound are given. The results of this paper can easily be extended over the ring $\mathbb{F}_q[u_1, u_2, \dots, u_r]/\langle f_1(u_1), f_2(u_2), \dots, f_r(u_r), u_i u_j - u_j u_i \rangle$ where polynomials $f_i(u_i)$, $1 \leq i \leq r$, split into distinct linear factors over \mathbb{F}_q .

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

Acknowledgements

The second author gratefully acknowledges support from Council of Scientific & Industrial Research (CSIR), India, Sanction No. 21(1042)/17/EMR-II.

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