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Rootless B-Fredholm Operators

Gabriel Kantún-Montiel[1*]

[1] *Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Av. Sn Claudio y 18 Sur Puebla 72570, Mexico.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Short Research Article

Abstract

Let T be a B-Fredholm operator on a Banach space X. We say that T is rootless if there is no bounded linear operator S and no positive integer $n \geq 2$ such that $T = S^n$. In this note, some properties of the uniform topological descent are used to give some sufficient conditions for B-Fredholm operators to be rootless.

Keywords: B-Fredholm operators; topological uniform descent; roots of operators.

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1 Introduction

Conditions for the existence of roots of operators are of interest in the study of algebraic properties of operators, and this problem have been investigated for several special classes of operators (see, for instance, [1], [2]). In the case of matrices, the roots problem has several applications such as the the matrix sign function, which is an useful tool in several problems (see, for instance [3], [4]).

Corresponding author: E-mail: gabriel.kantun@correo.buap.mx;

In [5] some conditions were given for a Fredholm operator to be rootless. For this, the index, ascent and descent play a key role. In this note, we prove a similar result for B-Fredholm operators. In this case, the uniform topological descent comes into play.

2 Preliminaries

Let X be a Banach space, and let us denote by $B(X)$ the set of all bounded linear transformations $T : X \to X$. Let $R(T)$ denote the range of T and $N(T)$ its null space. For $T, S \in B(X)$ the following holds:

$$R(TS) \subseteq R(T),$$
$$N(S) \subseteq N(TS).$$

In particular, defining $T^0 = I$, with $I \in B(X)$ the identity operator, we have the following chains:

$$X = R(T^0) \supseteq R(T) \supseteq R(T^2) \supseteq \cdots \supseteq R(T^n) \supseteq R(T^{n+1}) \supseteq \cdots$$

$$\{0\} = N(T^0) \subseteq N(T) \subseteq N(T^2) \subseteq \cdots \subseteq N(T^n) \subseteq N(T^{n+1}) \subseteq \cdots$$

The ascent of T, denoted by $a(T)$, and the descent of T, denoted by $d(T)$, are defined by:

$$a(T) = \inf\{n \in \mathbb{N}_0 : N(T^n) = N(T^{n+1})\},$$

$$d(T) = \inf\{n \in \mathbb{N}_0 : R(T^n) = R(T^{n+1})\},$$

with $\inf \emptyset = \infty$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Recall an operator T is Fredholm if $\alpha(T) := \dim N(T) < \infty$ and $\beta(T) = \dim X/R(T) < \infty$. For a Fredholm operator T, the index is $i(T) := \alpha(T) - \beta(T)$.

There are several ways to "generalize" Fredholm operators. In this paper we are concerned with B-Fredholm operators, which were introduced in 1999 by M. Berkani([6]): Let $T \in B(X)$ and $T_n : R(T^n) \to R(T^n)$ the operator given by $T_n x = Tx$. We say that T is a B-Fredholm operator if for some $n \in \mathbb{N}$ the range of T^n is closed and T_n is a Fredholm operator.

Let T be a B-Fredholm operator such that T_n is Fredholm, then T_m is Fredholm and $i(T_n) = i(T_m)$ for every $m \geq n$ ([6]). Thus, we can define the index of the B-Fredholm operator T as the index of the Fredholm operator T_n.

In a similar way as for Fredholm operators, if T is a B-Fredholm operator, then $i(T^n) = ni(T)$ for $n \in \mathbb{N}$. However, we don't have an analogue for the following: if S and T are Fredholm then TS is Fredholm and $i(TS) = i(T) + i(S)$ ([7]).

3 Topological Uniform Descent

Let T be a B-Fredholm operator, then T_n is a Fredholm operator for some $n \in \mathbb{N}$. Now, we are interested in $\alpha(T_n)$ and $\beta(T_n)$ with respect to $R(T^j)$ and $N(T^k)$ for $j, k = 0, 1, 2, \ldots$

Recall that if the ascent and descent of an operator $T \in B(X)$ are finite, then they coincide. Moreover, these quantities are related to $\alpha(T)$ and $\beta(T)$ as follows ([8, Theorem 3.4]):

- If $a(T) < \infty$ then $\alpha(T) \leq \beta(T)$;
- If $d(T) < \infty$ then $\beta(T) \leq \alpha(T)$;
- If $a(T) = d(T) < \infty$ then $\alpha(T) = \beta(T)$, but it can happen $\alpha(T) = \infty$;
- If $\alpha(T) = \beta(T) < \infty$ and if $a(T)$ or $d(T)$ is finite, then $a(T) = d(T)$.

Related to these quantities, we have the following ([9]):

$$\alpha_n(T) = \dim \frac{N(T^{n+1})}{N(T^n)};$$

$$\beta_n(T) = \dim \frac{R(T^n)}{R(T^{n+1})}.$$

Notice that $\alpha(T) = \alpha_0(T)$ and $\beta(T) = \beta_0(T)$.

Recall from [9, Lemma 3.1] that we have

$$\alpha_n(T) = \dim[N(T) \cap R(T^n)] = \alpha(T_n),$$

and by [9, Lemma 3.2]

$$\beta_n(T) = \dim \frac{X}{R(T) + N(T^n)} = \beta(T_n).$$

If T is of finite ascent, say $a(T) = n$, then $\alpha_n(T) = 0$. In a similar way, if T is of finite descent, then $\beta_n(T) = 0$. Now, following Grabiner ([10]) we are interested in the case when $\alpha_n(T)$ is not zero but constant: An operator $T \in B(X)$ is of topological uniform descent if there exists $d \in \mathbb{N}_0$ such that $N(T) \cap R(T^n) = N(T) \cap R(T^{n+1})$ and $R(T^n)$ is closed for all $n \geq d$. Equivalently, if $\alpha_n(T) = \alpha_{n+1}(T)$ for every $n \geq d$.

It is well known that B-Fredholm operators are of topological uniform descent ([6]). Here we offer a proof for the sake of completeness (see the proof of Proposition 2.6 in [6]) :

Theorem 3.1. *Let $T \in B(X)$ a B-Fredholm operator. Then T is of topological uniform descent.*

Proof. Since $R(T^m) \supseteq R(T^{m+1})$ we have that

$$N(T) \cap R(T^{m+1}) \subseteq N(T) \cap R(T^m).$$

Since T is B-Fredholm, there exists m such that $R(T^m)$ is closed and T_m is a Fredholm operator. It follows that $\dim N(T) \cap R(T^m) = \dim N(T_m) < \infty$. Hence, there exists $d \in \mathbb{N}$ such that $N(T) \cap R(T^m) = N(T) \cap R(T^d)$ for every $m \geq d$. That is, $\alpha_m(T) = \alpha_d(T)$ for $m \geq d$. Thus, T is of topological uniform descent. ☐

Now we prove the result that will enable us to study the "roots" of B-Fredholm operators.

Theorem 3.2. *Let $T \in B(X)$ be a B-Fredholm operator. Then, there exists $d \in \mathbb{N}_0$ such that $\alpha_d(T^n) = n\alpha_d(T)$ for every $n \geq d$.*

Proof. From the previous theorem, T is of topological uniform descent. Using [11, Lemma 2.1] repeatedly, we have that if $M_0 \subset M_1 \subset \cdots \subset M_n$ are subspaces of X, then

$$\dim \frac{M_n}{M_0} = \sum_{k=1}^{n} \dim \frac{M_k}{M_{k-1}}.$$

From

$$\alpha_d(T^n) = \dim \frac{N(T^{n(d+1)})}{N(T^{nd})} = \dim \frac{N(T^{nd+n})}{N(T^{nd})},$$

and

$$N(T^{nd}) \subseteq N(T^{nd+1}) \subseteq \cdots \subseteq N(T^{nd+n}),$$

it follows

$$\begin{aligned}\alpha_d(T^n) &= \sum_{j=1}^{n} \dim \frac{N(T^{nd+j})}{N(T^{nd+j-1})} \\ &= \sum_{j=1}^{n} \alpha_{nd+j-1}(T) \\ &= \sum_{j=1}^{n} \alpha_d(T) \\ &= n\alpha_d(T).\end{aligned}$$

□

The following known theorem illustrates the use of the results of this section:

Theorem 3.3. *Let $T \in B(X)$ be a Fredholm operator. If $\alpha(T) = 1$ and T is not of finite ascent, then $\alpha(T^n) = n$.*

Proof. Recall

$$\alpha_n(T) = \dim[R(T^n) \cap N(T)].$$

Since $R(T^m) \supseteq R(T^{m+1})$ for every $m \in \mathbb{N}$, we have

$$\alpha_0(T) \geq \alpha_1(T) \geq \cdots \geq \alpha_d(T) = \alpha_{d+1}(T) = \cdots,$$

where d corresponds to the topological uniform descent.

Since $\alpha_0(T) = \alpha(T) = 1$, then $\alpha_d(T) = 0$ or $\alpha_d(T) = 1$. But $a(T)$ is not finite, so $N(T^n) \neq N(T^{n+1})$ for every n, and

$$\dim \frac{N(T^{d+1})}{N(T^d)} \neq 0.$$

Hence, $\alpha_d(T) = 1$.

Now, since $\alpha_0(T) = \alpha_d(T)$, then

$$\alpha(T) = \alpha_d(T).$$

thus, by Theorem 3.2,

$$\alpha(T^n) = \alpha_d(T^n) = n\alpha_d(T) = n\alpha(T) = n.$$

□

4 Roots

Let $n \geq 2$, we say that $T \in B(X)$ have n-th root if there exists $S \in B(X)$ such that $T = S^n$. Of course, the study of the roots of an operator is deeply related to the properties of the ascent, descent, nullity and defect we have discussed so far. For instance, if $T \in B(X)$ is a Fredholm operator and $S \in B(X)$ is a root for T, then it is well known that S is a Fredholm operator.

Of course, T is rootless if there is no bounded linear operator S and no positive integer $n \geq 2$ such that $T = S^n$. We are especially interested in rootless B-Fredholm operators:

Theorem 4.1. *Let $T \in B(X)$ be a B-Fredholm operator. If $i(T) = \pm 1$, or if $\alpha_d(T) = 1$, then T is rootless.*

Proof. Suppose $i(T) = \pm 1$ and there exists $S \in B(X)$ and $n \in \mathbb{N}$ such that $T = S^n$. Since T is B-Fredholm, then S is B-Fredholm (see [6, Proposition 2.8]) and by Theorem 3.2 we have

$$i(T) = i(S^n) = ni(S).$$

Hence $ni(S) = \pm 1$, and it follows $n = 1$ and $T = S$. Therefore, T is rootless.

Now suppose $\alpha_d(T) = 1$ and $T = S^n$, for $S \in B(X)$, $n \in \mathbb{N}$. Since T is B-Fredholm we have

$$\alpha_d(T) = \alpha_d(S^n) = n\alpha_d(S).$$

Then $n\alpha_d(S) = \pm 1$, and it follows $n = 1$ and $T = S$. Therefore T is rootless. □

Since Fredholm operators are B-Fredholm, we recover a result from [5].

Corollary 4.2. *Let $T \in B(X)$ a Fredholm operator. If $\alpha(T) = 1$ and T is not of finite ascent, then T is rootless.*

Proof. From the proof of Theorem 3.3 we have $\alpha_d(T) = 1$, and from Theorem 4.1 it follows that T is rootless. □

Example 4.3. *Let$X = \ell_2$, the space of square summable sequences, and let us consider the operator $T \in B(X)$ defined by*

$$T(x_1, x_2, x_3, \cdots) = (x_1, x_3, x_4, x_5, \ldots).$$

Then $\alpha(T) = 1$, $R(T) = X$ and $\beta(T) = 0$. It follows $i(T) = 1$ and hence T is rootless.

5 Conclusion

B-Fredholm operators are a generalization of Fredholm operators. So far, several properties of Fredholm operators have found analogues for B-Fredholm operators. In this paper, we extended a result of Yood concerning roots of Fredholm operators to the case of B-Fredholm operators. We gave sufficient conditions for a B-Fredholm operators to be rootless using some properties of the uniform topological descent. In the future, more examples and applications will be investigated.

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Competing Interests

Author has declared that no competing interests exists.

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