Asian Research Journal of Mathematics

16(6): 30-36, 2020; Article no.ARJOM.56330

ISSN: 2456-477X

Optimal Bounds of the Arithmetic Mean by Harmonic, Contra-harmonic and New Seiffert-like Means

Hui-Zuo Xu[1*] and Wei-Mao Qian[2]

[1] *Teacher Development Center, Wenzhou Broadcast and TV University, Wenzhou 325000, China.*

[2] *School of Continuing Education, Huzhou Broadcast and TV University, Huzhou 313000, China.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i630195

Editor(s):

(1) Prof. Wei-Shih Du, National Kaohsiung Normal University, Taiwan.

Reviewers:

(1) Zhang Wenpeng, Northwest University, China.

(2) Abdullah Akkurt, Kahramanmara St mam University, Turkey.

Complete Peer review History: http://www.sdiarticle4.com/review-history/56330

Received: 24 February 2020

Accepted: 28 April 2020

Published: 19 May 2020

Original Research Article

Abstract

We provide the optimal bounds for the arithmetic mean in terms of harmonic, contra-harmonic and new Seiffert-like means.

Keywords: Seiffert-like mean; arithmetic mean; harmonic mean; contra-harmonic mean.

2010 Mathematics Subject Classification: 26D15, 26E60.

1 Introduction

Seiffert [1, 2] introduced two means

$$P(a,b) = \frac{a-b}{2\arcsin\left[(a-b)/(a+b)\right]},$$

$$T(a,b) = \frac{a-b}{2\arctan\left[(a-b)/(a+b)\right]}.$$

**Corresponding author: E-mail: huizuoxu@163.com;*

These two means are called the first and second Seiffert means, respectively.

For two positive and unequal real numbers a and b, Witkowski [3] introduced the Seiffert-like mean $M_f(a,b)$ given by the formula

$$M_f(a,b) = \frac{a-b}{2f\left[(a-b)/(a+b)\right]}, \tag{1.1}$$

where the function $f:(0,1)\mapsto\mathbb{R}$ (called Seiffert function) satisfying

$$\frac{x}{1+x} \le f(x) \le \frac{x}{1-x}.$$

It was shown that every symmetric and homogeneous mean of two positive real numbers can be represented in the form (1.1) and that every Seiffert function produces a mean. The correspondence between means and Seiffert functions is given by the formula

$$f(x) = \frac{x}{M_f(1-x,1+x)}, \text{ where } x = \frac{|a-b|}{a+b}. \tag{1.2}$$

Witkowski proved that the following conditions are equivalent:

$$M_f(a,b) < M_g(a,b) \Leftrightarrow f(x) > g(x). \tag{1.3}$$

The Neuman-Sándor mean $NS(a,b)$ and logarithmic mean $L(a,b)$ are the Seiffert-like means.

$$NS(a,b) = \frac{a-b}{2\text{arcsinh}\left[(a-b)/(a+b)\right]} := M_{\text{arcsinh}}(a,b),$$

$$L(a,b) = \frac{a-b}{2\text{arctanh}\left[(a-b)/(a+b)\right]} := M_{\text{arctanh}}(a,b),$$

Certainly, the first and second Seiffert means $P(a,b)$ and $T(a,b)$ can be denoted $M_{\text{arcsin}}(a,b)$ and $M_{\text{arctan}}(a,b)$. Further more, Witkowski extend the new Seiffert-like means by showing that also sine, tangent, hyperbolic sine and hyperbolic tangent are Seiffert functions, they are given as follows:

$$M_{\sin}(a,b) = \frac{a-b}{2\sin\left[(a-b)/(a+b)\right]},\ M_{\tan}(a,b) = \frac{a-b}{2\tan\left[(a-b)/(a+b)\right]}, \tag{1.4}$$

$$M_{\sinh}(a,b) = \frac{a-b}{2\sinh\left[(a-b)/(a+b)\right]},\ M_{\tanh}(a,b) = \frac{a-b}{2\tanh\left[(a-b)/(a+b)\right]}, \tag{1.5}$$

In recent years, these Seiffert-like means and their inequalities have attracted attention of several researchers [3, 4, 5, 6]. Undoubtedly, the Seiffert-like means are studied always compared with some well-known symmetric and homogeneous means of positive arguments.

Let $p\in\mathbb{R}$ and $a,b>0$ with $a\ne b$, the pth Hölder mean $H_p(a,b)$ are defined by

$$H_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\ne 0, \\ \sqrt{ab}, & p=0, \end{cases}$$

particularly

$$H_{-1}(a,b) = \frac{2ab}{a+b} := H(a,b),\ H_0(a,b) = \sqrt{ab} := G(a,b),$$

$$H_1(a,b) = \frac{a+b}{2} := A(a,b),\ H_2(a,b) = \sqrt{\frac{a^2+b^2}{2}} := Q(a,b),$$

are the harmonic mean, geometric mean, arithmetic mean and quadratic mean, respectively.

It is well-known that the Hölder mean $H_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b>0$ with $a \neq b$, hence the following inequalities hold

$$H(a,b) < G(a,b) < A(a,b) < Q(a,b) < C(a,b)$$

where $C(a,b) = (a^2+b^2)/(a+b)$ is contra-harmonic mean.

Let $a > b > 0$ and $x = (a-b)/(a+b) \in (0,1)$, Witkowski [3, Lemma 3.1-3.2] proved the following chains of inequalities

$$\operatorname{arctanh}(x) > \tan(x) > \sinh(x) > x > \operatorname{arcsinh}(x) > \sin(x) > \arctan(x) > \tanh(x)$$

hold for all $x \in (0,1)$. From (1.3) the following chains inequalities of means

$$\begin{aligned} &M_{\operatorname{arctanh}}(a,b) < M_{\tan}(a,b) < M_{\sinh}(a,b) < A \\ &< M_{\operatorname{arcsinh}}(a,b) < M_{\sin}(a,b) < M_{\arctan}(a,b) < M_{\tanh}(a,b) \end{aligned} \tag{1.6}$$

hold for $a,b>0$ with $a \neq b$.

From the formula (1.2), we can get the Serffert functions of the harmonic, geometric, arithmetic, quadratic and contra-harmonic means, they are listed as follows:

$$h(x) = \frac{x}{1-x^2},\ g(x) = \frac{x}{\sqrt{1-x^2}},\ a(x) = x,\ q(x) = \frac{x}{\sqrt{1+x^2}}. \tag{1.7}$$

Note that

$$h(0) - \operatorname{arctanh}(0) = 0,\ [h(x) - \operatorname{arctanh}(x)]' = \frac{2x}{(1-x^2)^2} > 0 \Leftrightarrow h(x) > \operatorname{arctanh}(x),$$

$$\cosh^2(x) > \left(1 + x^2/2\right)^2 > 1 + x^2 \Leftrightarrow \frac{1}{\cosh^2(x)} < \frac{1}{1+x^2} \Leftrightarrow \tanh(x) > \frac{x}{\sqrt{1+x^2}},$$

for $x \in (0,1)$.

Therefore,

$$H(a,b) < M_{\operatorname{arctanh}}(a,b),\ M_{\tanh}(a,b) < Q(a,b), \tag{1.8}$$

hold for all $a,b>0$ with $a \neq b$. From (1.6), (1.8) we obtain chains inequalities

$$\begin{aligned} &H(a,b) < M_{\operatorname{arctanh}}(a,b) < M_{\tan}(a,b) < M_{\sinh}(a,b) < A(a,b) \\ &< M_{\operatorname{arcsinh}}(a,b) < M_{\sin}(a,b) < M_{\arctan}(a,b) < M_{\tanh}(a,b) < Q(a,b) < C(a,b), \end{aligned} \tag{1.9}$$

Y.-M.Chu [7] et al. find the greatest value α and the least value β such that the double inequality

$$\alpha T(a,b) + (1-\alpha) G(a,b) < A(a,b) < \beta T(a,b) + (1-\beta) G(a,b), \tag{1.10}$$

hold for all $a,b>0$ with $a \neq b$.

F.Yang [8] et al. find the greatest value α and the least value β such that the double inequality

$$\alpha NS(a,b) + (1-\alpha) H(a,b) < A(a,b) < \beta NS(a,b) + (1-\beta) H(a,b), \tag{1.11}$$

hold for all $a,b>0$ with $a \neq b$.

Motivated by inequalities (1.9)-(1.11), we will present the best possible parameters $\alpha_i, \beta_i \in \mathbb{R}$ ($i = 1,2,3,4$) such that the double inequalities

$$\begin{aligned} &\alpha_1 C(a,b) + (1-\alpha_1) M_{\tan}(a,b) < A(a,b) < \beta_1 C(a,b) + (1-\beta_1) M_{\tan}(a,b), \\ &\alpha_2 C(a,b) + (1-\alpha_2) M_{\sinh}(a,b) < A(a,b) < \beta_2 C(a,b) + (1-\beta_2) M_{\sinh}(a,b), \\ &\alpha_3 M_{\sin}(a,b) + (1-\alpha_3) H(a,b) < A(a,b) < \beta_3 M_{\sin}(a,b) + (1-\beta_3) H(a,b), \\ &\alpha_4 M_{\tanh}(a,b) + (1-\alpha_4) H(a,b) < A(a,b) < \beta_4 M_{\tanh}(a,b) + (1-\beta_4) H(a,b) \end{aligned}$$

hold for all $a,b>0$ with $a \neq b$.

2 Lemmas

In order to prove our main results we need some lemmas, which we present in this section.

Lemma 2.1. *(See [5]) Let $-\infty < a < b < +\infty$, and let $f,\ g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *The function*

$$f(x) = \frac{\tan(x)-x}{(1+x^2)\tan(x)-x}$$

is strictly increasing from $(0,1)$ onto $(1/4, [\tan(1)-1]/[2\tan(1)-1])$.

Proof. Let $f_1(x) = \tan(x) - x, f_2(x) = (1+x^2)\tan(x) - x$. Then elaborated computations lead to

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x)-f_1(0)}{f_2(x)-f_2(0)},$$

$$\frac{{f_1}'(x)}{{f_2}'(x)} = \frac{\tan^2(x)}{2x\tan(x)+(x^2+1)(1+\tan^2(x))-1} := \varphi(x), \tag{2.1}$$

$$\begin{aligned}\varphi'(x) &= \frac{2\tan(x)(x^2\tan^2(x)-\tan^2(x)+x^2)}{(x^2\tan^2(x)+\tan^2(x)+2x\tan(x)+x^2)^2} \\ &= \frac{2\tan(x)(x^2-\sin^2(x))}{(x^2\tan^2(x)+\tan^2(x)+2x\tan(x)+x^2)^2\cos^2(x)} > 0,\end{aligned} \tag{2.2}$$

and

$$f\left(0^+\right) = 1/4, f\left(1^-\right) = [\tan(1)-1]/[2\tan(1)-1] = 0.2635\cdots. \tag{2.3}$$

Therefore, Lemma 2.2 follows easily from (2.1)-(2.3) and Lemma 2.1 . ☐

Lemma 2.3. *The function*

$$g(x) = \frac{\sinh(x)-x}{(1+x^2)\sinh(x)-x}$$

is strictly decreasing from $(0,1)$ onto $([\sinh(1)-1]/[2\sinh(1)-1], 1/7)$.

Proof. Let $g_1(x) = \sinh(x) - x, g_2(x) = (1+x^2)\sinh(x) - x$. Then elaborated computations lead to

$$g(x) = \frac{g_1(x)}{g_2(x)} = \frac{g_1(x)-g_1(0)}{g_2(x)-g_2(0)},$$

$$g_1'(x) = \cosh(x) - 1,\ g_2'(x) = 2x\sinh(x) + (x^2+1)\cosh(x) - 1, \tag{2.4}$$

$$\frac{{g_1}'(x)}{{g_2}'(x)} = \frac{{g_1}'(x)-{g_1}'(0)}{{g_2}'(x)-{g_2}'(0)}, \tag{2.5}$$

$$\begin{aligned}\frac{{g_1}''(x)}{{g_2}''(x)} &= \frac{\sinh(x)}{4x\cosh(x)+2\sinh(x)+(x^2+1)\sinh(x)} \\ &= \frac{1}{4x/\tanh(x)+2+(x^2+1)}.\end{aligned} \tag{2.6}$$

It follows from (2.4) C(2.6) and together with the fact that the function $x \mapsto x/\tanh(x)$ is positive and strictly increasing on $(0, 1)$, we clearly see that $g_1''(x)/g_2''(x)$ is strictly decreasing on $(0, 1)$. Note that

$$g\left(0^+\right) = \frac{1}{7},\ g\left(1^-\right) = [\sinh(1) - 1]/[2\sinh(1) - 1] = 0.1297\cdots. \tag{2.7}$$

Therefore, Lemma 2.3 follows easily from (2.7) and the monotonicity of $g(x)$. ☐

Lemma 2.4. *The function*

$$h(x) = \frac{x^2\sin(x)}{x - (1 - x^2)\sin(x)}$$

is strictly decreasing from $(0, 1)$ *onto* $(\sin(1), 6/7)$.

Proof. Let $h_1(x) = x^2\sin(x)$,$h_2(x) = x - (1 - x^2)\sin(x)$. Then simple computations lead to

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{h_1(x) - h_1(0)}{h_2(x) - h_2(0)},$$

$$h_1'(x) = 2x\sin(x) + x^2\cos(x), h_2'(x) = 1 + (x^2 - 1)\cos(x) + 2x\sin(x), \tag{2.8}$$

$$\frac{h_1'(x)}{h_2'(x)} = \frac{h_1'(x) - h_1'(0)}{h_2'(x) - h_2'(0)},$$

$$\frac{h_1''(x)}{h_2''(x)} = \frac{4x\cos(x) + 2\sin(x) - x^2\sin(x)}{-(x^2 - 1)\sin(x) + 4x\cos(x) + 2\sin(x)} = \frac{1}{1 + \phi(x)}, \tag{2.9}$$

where

$$\phi(x) = \frac{\sin(x)}{4x\cos(x) + 2\sin(x) - x^2\sin(x)} = \frac{1}{4x/\tan(x) + 2 - x^2}. \tag{2.10}$$

It is easy to verify the function $x \mapsto x/\tan(x)$ is positive and strictly decreasing on $(0, 1)$, which imply that the function $\phi(x)$ is increasing on $(0, 1)$. Follow from (2.8)-(2.9) lead to the conclusion that $h_1''(x)/h_2''(x)$ is strictly decreasing on $(0, 1)$.

Note that

$$h\left(0^+\right) = \frac{6}{7}, h\left(1^-\right) = \sin(x) = 0.8414\cdots. \tag{2.11}$$

Therefore, Lemma 2.4 follows easily from (2.11) and Lemma 2.1 together with the monotonicity of $h(x)$. ☐

Lemma 2.5. *The function*

$$k(x) = \frac{x^2}{x/\tanh(x) - (1 - x^2)}$$

is strictly increasing from $(0, 1)$ *onto* $(3/4, \tanh(1))$.

Proof. Let $k_1(x) = x^2$,$k_2(x) = x/\tanh(x) - (1 - x^2)$. Then elaborated computations lead to

$$k(x) = \frac{k_1(x)}{k_2(x)} = \frac{k_1(x) - k_1(0)}{k_2(x) - k_2(0^+)}, \tag{2.12}$$

$$\frac{k'_1(x)}{k'_2(x)} = \frac{2x\sinh^2(x)}{2x\cosh^2(x) + \cosh(x)\sinh(x) - 3x}, \tag{2.13}$$

Let $k_3(x) = 2x\sinh^2(x)$, $k_4(x) = 2x\cosh^2(x) + \cosh(x)\sinh(x) - 3x$, one has

$$\frac{k_1'(x)}{k_2'(x)} = \frac{k_3(x)}{k_4(x)} = \frac{k_3(x) - k_3(0)}{k_4(x) - k_4(0)},$$

$$\frac{k_3{}'(x)}{k_4{}'(x)} = \frac{2x\cosh(x)\sinh(x) + \sinh^2(x)}{2x\cosh(x)\sinh(x) + 2\sinh^2(x)} = 1 - \frac{1}{2x/\tanh(x) + 2}. \tag{2.14}$$

By (2.14) and the function $x \mapsto x/\tanh(x)$ is positive and strictly increasing on $(0,1)$, we clearly see that $k_1'(x)/k_2'(x)$ is strictly increasing on $(0,1)$. Note that

$$k\left(0^+\right) = \frac{3}{4}, k\left(1^-\right) = \tanh(1) = 0.7615\cdots. \tag{2.15}$$

Therefore, Lemma 2.5 follows easily from (2.15) and Lemma 2.1 together with the monotonicity of $k(x)$. □

3 Main Results

Theorem 3.1. *The double inequalities*

$$\alpha_1 C(a,b) + (1-\alpha_1) M_{\tan}(a,b) < A(a,b) < \beta_1 C(a,b) + (1-\beta_1) M_{\tan}(a,b), \tag{3.1}$$

$$\alpha_2 C(a,b) + (1-\alpha_2) M_{\sinh}(a,b) < A(a,b) < \beta_2 C(a,b) + (1-\beta_2) M_{\sinh}(a,b), \tag{3.2}$$

hold for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \le 1/4$, $\beta_1 \ge [\tan(1) - 1]/[2\tan(1) - 1] = 0.2635\cdots$, $\alpha_2 \le [\sinh(1) - 1]/[2\sinh(1) - 1] = 0.1297\cdots$ *and* $\beta_2 \ge 1/7$.

Proof. Since all the bivariate means concerned in Theorem 3.1 are symmetric and homogeneous of degree one, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$. Then we making use of (1.4)- (1.5) and (1.7) lead to the conclusion that inequalities (3.1) and (3.2) are respectively equivalent to

$$\alpha_1 < \frac{A(a,b) - M_{\tan}(a,b)}{C(a,b) - M_{\tan}(a,b)} = \frac{\tan(x) - x}{(1+x^2)\tan(x) - x} := f(x) < \beta_1, \tag{3.3}$$

$$\alpha_2 < \frac{A(a,b) - M_{\sinh}(a,b)}{C(a,b) - M_{\sinh}(a,b)} = \frac{\sinh(x) - x}{(1+x^2)\sinh(x) - x} := g(x) < \beta_2, \tag{3.4}$$

where $f(x)$ and $g(x)$ are defined as in Lemmas 2.2 and 2.3.

Therefore, Theorem 3.1 follows easily from (3.3), (3.4) together with Lemmas 2.2 and 2.3. □

Theorem 3.2. *The double inequalities*

$$\alpha_3 M_{\sin}(a,b) + (1-\alpha_3) H(a,b) < A(a,b) < \beta_3 M_{\sin}(a,b) + (1-\beta_3) H(a,b), \tag{3.5}$$

$$\alpha_4 M_{\tanh}(a,b) + (1-\alpha_4) H(a,b) < A(a,b) < \beta_4 M_{\tanh}(a,b) + (1-\beta_4) H(a,b), \tag{3.6}$$

hold for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_3 \le \sin(1) = 0.8414\cdots$, $\beta_3 \ge 6/7$, $\alpha_4 \le 3/4$ *and* $\beta_4 \ge \tanh(1) = 0.7615\cdots$.

Proof. Since all the bivariate means concerned in Theorem 3.2 are symmetric and homogeneous of degree one, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$. Then we making use of (1.4)- (1.5) and (1.7) lead to the conclusion that inequalities (3.5) and (3.6) are respectively equivalent to

$$\alpha_3 < \frac{A(a,b) - H(a,b)}{M_{\sin}(a,b) - H(a,b)} = \frac{x^2\sin(x)}{x - (1-x^2)\sin(x)} := h(x) < \beta_3, \tag{3.7}$$

$$\alpha_4 < \frac{A(a,b) - H(a,b)}{M_{\tanh}(a,b) - H(a,b)} = \frac{x^2}{x/\tanh(x) - (1-x^2)} := k(x) < \beta_4, \tag{3.8}$$

where $h(x)$ and $k(x)$ are defined as in Lemmas 2.4 and 2.5.

Therefore, Theorem 3.2 follows easily from (3.7), (3.8) together with Lemmas 2.4 and 2.5. □

4 Conclusion

In this paper, we used mathematical analysis method and the monotonicity of the functions to study the arithmetic mean of some Seiffert-like functions, and obtained some optimal bounds of these arithmetical means.

Acknowledgment

This work was supported by the Key Project of the Scientific Research of Zhejiang Open University in 2019 (Grant no. XKT- 19Z02) and the "312 project of talents cultivation" of Zhejiang Open University (Grant No.7).

Competing Interests

The authors declare that they have no competing interests.

References

[1] Seiffert HJ. Werte zwischen dem geometrischen und dem arithmetischen Mittel zweier Zahlen. Elem. Math. 1987;42(4):105-107.

[2] Seiffert HJ. Aufgabe β 16. Die Wurzel. 1995;29(9+10):221-222.

[3] Witkowski A. On Seiffert-like means. J. Math. Inequal. 2015;4(9):1071-1092.

[4] Nowicka M, Witkowski A. Optimal bounds for the tangent and hyperbolic sine means, Aequat. Math; 2020.
Available:https://doi.org/10.1007/s00010-020-00705-6

[5] Nowicka M, Witkowski A. Optimal bounds for the tangent and hyperbolic sine means. J. Math. Inequal. 2020;14(1):23-32.

[6] Nowicka M, Witkowski. Optimal bounds for the arithmetic mean in terms of new Seiffert-like means. Math. Inequal. Appl. 2020;23(1):383-392.

[7] Chu YM, Zong C, Wang GD. Optimal convex combination bounds of Seiffert and geometric means for the arithmetic mean. J. Math. Inequal. 2011;5(3):429-434.

[8] Yang F, Chu YM, Qian WM. Bounds for the arithmetic mean in terms of the neuman-sndor and other bivariate means. Journal of Applied Mathematics. 2013;7. Article ID 582504
Availabe:http://dx.doi.org/10.1155/2013/582504

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/56330