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The Uniqueness of Stationary Solution for Nonlinear Random Reaction-diffusion Equation

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Authors' contributions

This work was carried out in collaboration between both authors. Author SH designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author SH managed the analyses of the study. Author FK managed the literature searches. Both authors read and approved the final manuscript.

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ABSTRACT

We study a nonlinear random reaction-diffusion problem in abstract Banach spaces, driven by a real noise, with random diffusion coefficient and random initial condition. We consider a polynomial non linear term. The reaction-diffusion equation belongs to the class of parabolic stochastic partial differential equations. We assume that the initial condition is an element of Hilbert space. The real noise is a Wiener process. We construct a suitable stochastic basis and define the solution of reaction-diffusion problem in the weak sense. We define the stationary process in abstract Banach spaces in the strong sense of Doob-Rozanov. That is, the probability density function of the stochastic process is independent of time shift. We define the invariant measure for random reaction-diffusion equation in the sense of Arnold, DaPrato, and Zabczyk [1,2]. In other words, we define the invariant measure for random dynamical system, associated with random reaction-diffusion problem.

Using the Variation Inequalities Theory, we prove the uniqueness of stationary solution for nonlinear random reaction-diffusion problem. The obtained theoretical results have several applications in Quantum Physics, Biology, Medicine, and Economic Sciences. Especially, we can study the existence of stationary solution for the stochastic models of tumor growth.

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1. INTRODUCTION

Random reaction-diffusion equations form an important part of the theory of random partial differential equations that is both very rich changelings mathematically and is related in physics, chemistry, biology, medicine, astronomy, and Economic Sciences. To be specific, in this paper we consider a random reaction-diffusion equation with a polynomial nonlinearity. Of course, we can consider more general mathematical models. For example, we can investigate the random nerve equations, random Lotka- Volterra equations, random Boussinesq- Glover equations, random superfluid equations, random Belousov-Zhabotinsky reaction equations in chemical dynamics, etc.

Let $(\Omega, \mathrm{F}, (\mathrm{F_t})_{t\geq 0}, P)$ be a stochastic basis, and let $w(t) = (w_1(t), w_2(t), \dots\dots, w_m(t))$ be a standard m-dimensional Wiener process defined on $(\Omega, \mathrm{F}, (\mathrm{F_t})_{t\geq 0}, P)$. Let $\hat{\xi}(t)$ be a stationary solution of the Ito equation in R^k

$$d\xi(t) = a\xi(t)dt + b\xi(t)dw(t),\ t \geq 0, \qquad (1)$$

where $\mathrm{a}(\cdot)$ and $\mathrm{b}(\cdot)$ satisfy the assumptions of section 2.1 below. We look at the process $(\Omega, \mathrm{F}, (\mathrm{F_t})_{t\geq 0}, P, (\xi(t))_{t\geq 0})$ as a model of real noise, stationary in time. Having assumed that the noise process is given, we consider the random nonlinear evolution equation in Hilbert spaces driven by the real noise $\hat{\xi}(t)$:

$$\frac{du(t,\omega)}{dt} + A\left(\hat{\xi}(t,\omega), u(t,\omega)\right) = f\left(\hat{\xi}(t,\omega)\right), t \geq 0,\ \ \omega \in \Omega, \qquad (2)$$

Where $\{A(\xi,\cdot), \xi \in R^k\}$ is a family of monotone operators in a Gelfand triplet $\subset H \subset V^{'}$,

and f is a function from $R^k\ to\, V^{'}$ (see section 2.2.1 for detailed assumptions on A and f). The aim of this paper is to prove the existence of a stationary solution of equation (2). Note that this equation does not contain Ito differential.

In the last few years a lot of papers appeared on invariant measures and stationary solutions for Ito type equations in Hilbert spaces. The case of real noise is not treated. In comparison with the existing literature, we mention two aspects of this paper.

The first one is that we want to consider the real noise $\hat{\xi}(t)$ as a given Markov process, stationary in time. Corresponding to this process$\hat{\xi}(t)$, we would find a stationary solution of equation (2). This fact motivates some technical details of the following analysis, like the choice of a special stochastic basis (see section 3.1) and Theorem 3.1, which are novel with respect to the literature conceding with Ito equations.

The second point is that we shall not assume any compactness. At our knowledge, all the methods know in the literature to prove the existence of invariant measures or stationary solutions use some compactness, coming from the topologies of the function spaces involved. It is well-know that the structure of the monotonicity allows to prove existence of solutions without any compactness assumption.

At the end of the paper we give some applications, which contributed to motivate our analysis.

2. EXISTENCE OF A STATIONARY SOLUTION

2.1 Definition of Stationary Solution

We introduce now the stochastic basis necessary for the sequel. Let us choose an invariant measure m_0 for equation $d\xi(t) = a\big(\xi(t)\big)dt + b\big(\xi(t)\big)dw(t),\ \ t \geq 0$ with finite moments of every order, or at least with finite moment of order r, with r given by assumption (A.4) there are a function $\rho: R^k \to [0,\infty)$ and constants $p \geq 2,\ \lambda \in R,\ \alpha > 0,\ r \geq 0, C_\rho > 0,$ such that for every $\xi \in R^k, u \in V$,

$$2 \quad \langle A(\xi,u),u\rangle + \lambda|u|^2\rho(\xi) \geq \alpha\|u\|^p, \quad \text{and} \quad \rho(\xi) \leq C_\rho(1 + |\xi|^r_{R^k}).$$

Let $(\Omega^0, F^0, (F_t^0)_{t\geq 0}, P^0)$ be any stochastic basis supporting a m-dimensional standard Wiener process $w(t)$. Let

$$\Omega = R^k \times [0,1] \times \Omega^0, F = B(R^k)\otimes B(0,1)\otimes F^0,$$

$$\mathrm{F_t} = B(R^k)\otimes B(0,1) \otimes F_t^0\ , P = m_0\otimes\lambda\otimes P^0 \ (3)$$

where B stands for the Borel σ −algebra , and λ is the Lebesgue measure on [0,1].

Consider $w(t)$ as a Brownian motion on the new stochastic basis $(\Omega, F, (F_t)_{t\geq 0}, P)$.

In other words, define

$$\widehat{w}(t,x,y,\omega^0) = w(t,\omega^0), \quad x \in R^k, \quad y \in [0,1], \quad \omega^0 \in \Omega^0$$

then denote $\widehat{w}(t)$ simply by $w(t)$.

Let $\widehat{\xi_0} : \Omega \to R^k$ be defined as

$$\widehat{\xi_0}(x,y,\omega^0) = x, \; x \in R^k, y \in [0,1], \; \omega^0 \in \Omega^0 \quad (4)$$

Then law of $\widehat{\xi_0}$ is m_0. By the assumption of the r-th moment of m_0, we have $\widehat{\xi_0} \in L^r(\Omega, F_0, P, R^k)$. Since m_0 is an invariant measure, and equation (3) defines a Markov process, the solution$\hat{\xi}(t)$ of equation $d\xi(t) = a(\xi(t))dt + b(\xi(t))dw(t), \quad t \geq 0$, with the initial condition $\widehat{\xi_0}$ is a stationary process. We have used the componentR^k of the stochastic basis to construct $\widehat{\xi_0}$ and the stationary solution $\hat{\xi}(t)$. The component [0,1] of the stochastic basis will be used to construct suitable initial conditions for the Galerkin approximations of the monotone equation (equation with monotone operator). Given the stochastic basis and the stationary process $u(t,\omega)$ just defined , we say that a stochastic process $\hat{\xi}(t$ in the sense of the previous subsection, with respect to some initial condition u_0 satisfying (10), and in addition u is a stationary process in H. This means that for all $n \in N, \; 0 \leq t_1 < t_2 < \cdots < t_n \, and \, h > 0$, and joint law of random element

$$(u(t_1+h), u(t_2+h), \ldots, u(t_n+h) \in H^n$$

is independent of h.

Remark 3. The definition given above corresponds to the viewpoint that the noise is a given process. So, in a sence, we look for a form of strong solutions, instead of weak solutions. However, we choose a suitable stochastic basis from the beginning so that our concept is in betweens from weak and strong solutions, see [3,4].

A stronger version of the previous would require that the joint process $(\hat{\xi}(t), u(t))$ is stationary in $R^k \times H$. In fact, this is the form of stationarity that we shall prove.

3. MAIN RESULTS

Call the definition of β .Let $\beta > 0$ be a constant satisfying the (Poincare type) inequality

$$|v|^2 \leq \beta \|v\|^2, \quad \forall v \in V$$

Theorem 3.1

Asume conditions:

(A.1) for every $\in V$, $\xi \to A(\xi,u)$ is a strong measurable mapping from R^k to V' ,bounded in bounded sets,

(A.2) for every $\xi \in R^k \quad u, v, z \in V$, the function

$s \to \langle A(\xi, v+sz), u\rangle$ is continuous on R.

On the stochastic basis (3) consider the random variable $\widehat{\xi_0}$ defined by (4) and the associated stationary solution $\hat{\xi}(t)$ of equation $d\xi(t) = a(\xi(t))dt + b(\xi(t))dw(t), \quad t \geq 0$,.

Under the hypotheses

(A.1) for every, $\xi \to A(\xi,u)$ is a strong measurable mapping from R^k to V', bounded in bounded sets,

(A.2) for every $\xi \in R^k \quad u, v, z \in V$, the function

$s \to \langle A(\xi, v+sz), u\rangle$ is continuous on R.

(A.3) there is a constant $\lambda_0 \in R$ such that for every $\in R^k$, $u, v \in V$

$$2\langle A(\xi,u) - A(\xi,v), u-v\rangle + \lambda_0 |u-v|^2 \geq 0,$$

(A.4) there are a function $\rho: R^k \to [0,\infty)$ and constants $p \geq 2, \; \lambda \in R, \; \alpha > 0, \; r \geq 0, C_\rho > 0$, such that for every $\xi \in R^k, u \in V$,

$$2 \quad \langle A(\xi,u), u\rangle + \lambda |u|^2 \rho(\xi) \geq \alpha \|u\|^p, \quad \text{and} \quad \rho(\xi) \leq C_\rho(1 + |\xi|_{R^k}^r)$$

(A.5) with p as above, there is a constant $C_A > 0$ such that for every $\in R^k$, $u \in V$,

$$\|A(\xi,u)\|_{V'} \leq C_A(1 + \|u\|^{p-1})$$

Finally, let $f = f(\xi): R^k \to V'$ be a given strong measurable function which satisfies the assumption (f.1) there is constant $C_f > 0$, such that for all $\xi \in R^k$

$$\|f(\xi)\|_{V'}^{p'} \leq C_f(1 + |\xi|_{R^k}^r)$$

Where p' is the conjugate exponent of p (i.e $\frac{1}{p} + \frac{1}{p'} = 1$) suppose that $\lambda < \frac{\alpha}{\beta}$.

Then equation $\frac{du(t,\omega)}{dt} + A(\xi(t,\omega)$, u($t,\omega)) = f(\xi(t,\omega))$, $t \geq 0$, $\omega \in \Omega$, with the initial condition $u(0,\omega) = u_0(w)$, $\omega \in$, satisfying $u_0 \in L^2(\Omega, \mathrm{F}_0, P, H)$, has a stationary solution.

Proof

Step 1.Invariant measure for the approximating problem

For each given natural number n, let as consider

$$\frac{du_n(t,\omega)}{dt} + A_n(\xi(t,\omega), u_n(t,\omega)) = f_n(\xi(t,\omega)), t \geq 0, \;\; \omega \in \Omega \quad (5)$$

$$d\xi(t) = a(\xi(t))dt + b(\xi(t))dw(t)\,,\;\; t \geq 0 \quad (6)$$

With initial conditions

$$u_n(0,\omega) = u_{0n}(w), \quad \xi(0,\omega) = \xi_0(\omega) \quad (7)$$

on the stochastic basis (3). In (7) choose as ξ_0 the random variable $\widehat{\xi_0}$ defined by (4). Moreover, choose$u_{0n} = 0$ for all n.

Corresponding to these initial conditions, let $(u_n(t), \hat{\xi}(t)$) be the solution of (5)-(6). Denote by u_t^n the law of $(u_n(t), \hat{\xi}(t)\; on\; H \times R^k\; and\; by\; \rho_1^n\; the\; law\; on\; H \times R^k$ defined as

$$\rho_1^n = \frac{1}{t}\int_0^t \mu_s^n \, ds \quad (8)$$

Both μ_t^n and ρ_1^n have marginal $m_0\; on\; R^k$. The family$\{\mu_t^n\;,\; t \geq 0\}$is tight. To prove this, in the inequality

$$|u_n(t,\omega)|^2 \leq e^{\gamma t}|u_{0n}(\omega)|^2 + \int_0^1 e^{\gamma(t-s)} \cdot [a - \varepsilon + (C_f C(\varepsilon) + C_\rho)(1 + |\xi(s)|_{R^k}^r)] \cdot ds\,, \;\text{ for all } n \in N\,, t > 0\,, P - a.e.\omega \in \Omega \quad (9)$$

take $\varepsilon > 0$ so small that $\gamma < 0$. Note that

$E\left|\hat{\xi}(s)\,\right|_{R^k}^r$ is constant.

Hence

$$\sup_{t>0} E\int_0^t e^{\gamma(t-s)\cdot(1+\left|\hat{\xi}(s)\,\right|_{R^k}^r)} \cdot ds < \infty$$

we obtain

$$E|u_n(t)|^2 \leq C_1, \quad \forall n \in N\;,\; \forall t \geq 0 \quad (10)$$

for some constant $C_1 > 0$ indenpendent of n and t. The bound (10), along with the uniform bound for $E\left|\hat{\xi}(s)\,\right|_{R^k}^r$, imply by Chebishev that the family of measures $\{\mu_t^n\;,\; t \geq 0\}$ is tight. In fact the family

$\{\mu_t^n\;,\; t \geq 0, n \in N\}$ is tight, but it Isere not easy to use this additional information for a more direct proof of the Theorem.

Since we have proved that the family $\{\mu_t^n, \; t \geq 0\}$is tight, the family$\{\rho_t^n\;,\; t \geq 0\}$ is also tight. By Prohorov theorem, applied for every given n, there exists a probability measure μ^n on $H \times R^k$ that is weak limit of some sequence $\{\rho_{tj}^n\}_{j\in N}$.Clearly, μ^nhas marginal m_0 on R^k.

By a classical argument (see [5]) and based on the Markov and Feller property for system (13),(14) we can prove that μ^n is an invariant measure for this system. We have used the Krylov-Bogoliubov method, except that we have stressed the fact that all measures μ^n have the same marginal m_0.

Finally, we notice that there exists a constant C_2 independent of n, such that

$$\int_{H\times R^k}(|h|^2 + |x|^2)d\mu^n((h,x) \leq C_2\;,\;\; \forall n \in N \quad (11)$$

To prove this inequality, let $\{\theta_i(h,x)\}_{i\in N}$ be a sequence of continuous bounded function $(h,x) \in H \times R^k$, which converges as $i \to \infty$, from below, to the function

$(h,x) \to |h|^2 + |x|^2$ defined on $H \times R^k$

We have

$$\int_{H\times R^k} \theta_i(h,x)d\mu^n((h,x)) = \lim_{j\to\infty} \theta_i(h,x)d\rho_{tj}^n((h,x))$$
$$= \frac{1}{t}\int_0^1 E(|u_n(s)|^2)ds + E(\left|\hat{\xi}(s)\,\right|_{R^k}^2) \leq C_2$$

for some constant C_2 independent of n and j (this follows from (10) and the proposition that $E(\left|\hat{\xi}(s)\,\right|_{R^k}^2$ is constant)

Summarizing, we prove that there exists an invariant probability measure μ^n for the system (14)-(15), with marginal m_0 on R^k , satisfying the inequality (4).

Step 2. Stationary solutions for the approximating system

Again, we consider the system (5)-(6) for each given n separately. On the stochastic basis (3) (in fact, it is sufficient to use the component $R^k \times [0,1]$ of the basis), for each n there exists an F_0 measurable random variable $\hat{u}_{0,n}$ such that $(\widehat{\xi_0}, \hat{u}_{0,n})$ has joint law μ^n. This follows from Proposition 3.1 below:in the assumptions of the Proposition we take $\mu = \mu^n$ .Then, the Propositions gives the random variables ψ and ϕ , and we denote ψ by $\hat{u}_{0,n}$, so that $\phi = (\widehat{\xi_0}, \hat{u}_{0,n})$ (recall definition (4) of $\widehat{\xi_0}$)

By the Markov property, the solution $(\hat{\xi}(t), \hat{u}_n(t))$ of the system corresponding to the initial condition $(\widehat{\xi_0}, \hat{u}_{0,n})$ is a stationary process in $R^k \times H$

Finally, we have

$$E\left(\left|\hat{u}_{0,n}\right|^2\right) \leq E(|\hat{u}_0|^2) \leq C_2, \ \forall n \in N \tag{12}$$

Step 3. Limiting procedure

Let as now study the convergence, as$n \to \infty$, of the interval [0,T]. Let as apply part (iii) of Lemma 2.1 with initial conditions $(\widehat{\xi_0}, \hat{u}_{0,n})$ found in the previous step. Therefore, there is a subsequence $\hat{u}_{n_k}(t)$, still denoted by $\hat{u}_n(t)$ for simplicity of notation, such that

$\hat{u}_n(t) \to \hat{u}$ weakly in $L^P(\Omega \times [0,T]; V)$

$\hat{u}_n(t) \to \hat{u}$ weak-star in $L^2(\Omega; L^\infty(0,T;H))$

$A_n(\hat{\xi}, \hat{u}_n) \to A_n(\hat{\xi}, \hat{u}\ )$ weakly in $L^{P'}(\Omega \times [0,T]; V')$

$\frac{d\hat{u}_n(t)}{dt} = \frac{d\hat{u}(t)}{dt}$ weakly in $L^{P'}(\Omega \times [0,T]; V')$

$\hat{u}_n(0) \to u_0$ weakly in $L^2(\Omega; H)$

$\hat{u}_n(T) \to \hat{u}(T)$ weakly in $L^2(\Omega; H)$

Using conditions (A.2)-(A.4) and the classical monotonicity argument of [6] or [5]. A fortiori we can prove that

$$\hat{u} \in L^2(\Omega, C([0,T]; H)) \cap L^P(\Omega \times [0,T]; V)$$

by classical arguments. The continuity of paths can be proved path by path, using the deterministic technique as in [7].

By a diagonal procedure, the construction of the weakly convergent subsequence can be performed in such a way that the subsequence converges over each interval [0,T] to the same limiting process u.

In order to prove the stationary of the solution$\hat{u} = \hat{u}(t,\omega)$ we need (19) and the proposition

$\hat{u}_n \to \hat{u}$ weak-star in $L^2(\Omega, L^\infty(0,T;H))$

If

$P_n = law\ \hat{u}_n$ in $L^2(\Omega, L^\infty(0,T;H))$

$P = law\ \hat{u}$ in $L^2(\Omega, L^\infty(0,T;H))$

then (by Prohorov Theorem) there exists a subsequence $\{P_{n_k}\}$of the sequence P_n such that $P_{n_k} \to P\ as\ k \to \infty$ (in the weak sense). The proof is complete.

Let $R^k, [0,1]$, and H be endowed with Borel σ-algebras, and let ʎ be the Lebesgue measure on [0,1]. The next Proposition 3.1, used above in Step 3 deals with random variables from $R^k \times [0,1]$, to $R^k \times H$. It is assumed to have the product measure $m_0 \otimes \lambda$ on $R^k \times\ [0,1]$,. Given a measure $\mu\ on R^k \times H$, we can always find a random variable.

$$\phi \to R^k \times\ [0,1] \to R^k \times\ H$$

such that $\phi(m_0 \otimes \lambda\ )= \mu$. The Proposition 3.1 assert that, if μ has marginal$m_0\ on\ R^k$ one cane choose $\boldsymbol{\phi}$ as a special form. The proof is adapted from Skorohod Representation Theorem (see [8], pp. 9-10 and [9])

Proposition 3.1 Given the spaces $R^k, [0,1]$, and H be endowend with Borel σ- algebras, given a measure $m_0\ on\ R^k$, the Lebesgue measure ʎ on [0,1], and a measure μ on $R^k \times H$ with marginal $m_0\ on\ R^k$. Then, there exists a random variable.

$$\psi: R^k \times [0,1] \to H$$

such that, denoted by

$$\phi: R^k \times\ [0,1] \to R^k \times H$$

the mapping

$$\phi(x,y) = \big(x, \psi(x,y)\big), \quad x \in\ R^k, \quad y \in\ [0,1]$$

we have

$$\phi(m_0 \otimes \lambda\ )= \mu.$$

3.1 Sum of Monotone Operators

It is useful to generalize Theorem 3.1 in the case of finite sum of monotone operators. For each $i = 1,2,\dots,\upsilon$ we assume to have a real reflexive separable Banach space $V_i \subset H$ (norm $\|\cdot\|_{i)}$, densely embedded into H, with the usual identification.

$$V_i \subset H \subset V_i^{'}$$

Moreover, we assume to have a family of nonlinear operators from V_i to $V_i^{'}$, denoted by $A_i(\xi,\cdot)$, $\xi \in R^k$, satisfying assumptions (A.1)–(A.5) with constants $\lambda_0, p_i, \lambda_i, \alpha_i, r_i, C_p, C_{A_i}$ possibly depending on i. Finally, we assume to have strongly measurable functions.

$$f_i = f_i(\xi): R^k \to V_i^{'}$$

statisfyng assumption (f.1)

We consider the random evolution equation

$$\frac{du(t,\omega)}{dt} + \sum_{i=1}^{m} A_i(\hat{\xi}\,(t,\omega), u(t,\omega)) = \sum_{i=1}^{\upsilon} f_i\left(\hat{\xi}(t,\omega)\right), \quad t \geq 0, \;\; \omega \in \Omega \qquad (13)$$

where $\hat{\xi}(t,\omega)$, denotes the solution of equation $d\xi(t) = a\xi(t)dt + b\xi(t)dw(t), \quad t \geq 0$, defined in section 3.1, on the stochastic basis (Ω, F, F_t, P) given by (3). The definition of the stationary solution is the same as the one given in section 3.1. Let $\beta_i > 0$ be a constant satisfying the inequality

$$|x|^2 \leq \beta_i \|x\|_i^2 \quad , \;\forall x \in V_i, \qquad i = 1,2,\dots,\upsilon$$

Theorem 3.2 Assume that

$$\sum_{i=1}^{\upsilon} (\lambda_i - \frac{\alpha_i}{\beta_i}) < 0$$

Then the equation (32)(6) has a stationary solution. The proof in the same as the one of Theorem 3.1

4 APPLICATIONS IN NON-LINEAR RANDOM REACTION – DIFFUSION EQUATION WITH REAL NOISE

In this section study the random reaction-diffusion equation with polynomial nonlinearity

$$\frac{\partial u}{\partial t} = c\Delta u - \sum_{h=0}^{2p-1} a_h(x,\xi) u^k \qquad (14)$$

$x \in D \subset R^3$, $t \geq 0$, $p \in N$, where D denotes an open bounded set of R^3. The associated initial and boundary conditions are

$$u(0,x,\omega) = u_0(x,\omega) \qquad (15)$$

And

$$u(t,x,\omega) = 0\,, \quad x \in \partial D \qquad (16)$$

Assume that the diffusion coefficient $c = c(\xi)$ is no negative, bounded, measurable in ξ , and independent on x.

Assume that the random function $a_h(x,\xi)$, are measurable in (x,ξ) and there are the real constant $c_h < C_h$ and positive constants $c_{2p-1} < C_{2p-1}$ such that

$$c_h \leq a_h(x,\xi) \leq C_h, \qquad for\; 1 \leq h \leq 2p-1 \; and\; for\; each (x,\xi) \in D \times R^k$$

On the random function $a_0(x,\xi)$ we may impose the conditions:

$x \to a_0(x,\xi)$ is measurable in ξ ,belongs to $H^{-1}(D)$, and satisfies the condition:

$$|a_0(x,\xi)|_{H^{-1}(D)} \leq C(1 + |\xi|_{R^k}^r)$$

for some positive constants C and r

Assume that the initial conditions $u_0(x,\omega)$ satisfies $u_0 \in L^2(\Omega, \mathrm{F}_0, P, H)$

Thanks to the previous assumptions and Young inequality, there is constant $\bar{c} > 0$ such that

$$(\sum_{h=0}^{2p-1} a_h(x,\xi)u^h)u \geq \frac{1}{2} C_{2p-1} u^{2p} - \bar{c}$$

For the abstract formulation of the problem (33)(7), (34)(8), (35)(9), let $H = L^2(D)$, $V = H_0^1(D)$

$$V^{'} = H^{-1}(D), \quad A(\xi,u) = \sum_{h=0}^{2p-1} a_h(x,\xi)u^h \;, \qquad f(\xi) = -a_0\,(x,\xi),$$

The proof of the conditions (A.1)-(A.5) and (f.1) is classical. As to the disiparivity condition, we have $\gamma = 0\,, \delta = 0 \quad , a = 1/2C_{2p-1}$

Thus, we can apply Theorem 3.1 and obtain.

Theorem 4.1

Under the previous assumptions, the nonlinear random reaction diffusion problem (14), (15), (16) ad mints an unique stationary solution.

Corollary 4.1

Under the suitable assumptions, we can prove a similar result for the random Hodgkin-Huxley equations, Fitz –Hugh-Nagumo equations, Lotka-Voltera equations, Boussinesq –Glover equation, heat equation, Belousov- Zhabotinsky equations in chemical dynamics. For more details regarding to the random versions of above mentioned equations, see Chung [10], Cordoca and Bras [11], Flandoli and Kolaneci [9], Freeze [12], Kolaneci [13], Kutler [14], Murray [15], Sagar [4], Stengel [16] and [17-20].

5. CONCLUSION

In this paper we investigate uniqueness of stationary solution for nonlinear random reaction-diffusion equation in Banach spaces, driven by a real noise. We assume that diffusion coefficient is a random variable and the initial condition is a random function. The real noise process is defined as a stationary solution of ito stochastic differential equation in finite dimensional Euclidian space or Hilbert space. To be specific, we consider a random reaction – diffusion equation with a polynomial nonlinearity. Of course, we can investigate more general mathematical models, and suggest several applications, especially Boussinesq –Glover equation.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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