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Boundedness of commutators on herz-morry-hardy spaces with variable exponent

Omer Abdalrhman[1,2*], Afif Abdalmonem[1,3] and Shuangping Tao[1]

1 College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, P.R. China.; afeefy86@gmail.com(A.A); taosp@nwnu.edu.cn(S.T).

2 College of Education, Shendi University, Shendi, River Nile State, Sudan.

3 Faculty of Science, University of Dalanj, Dalanj, South kordofan, Sudan.

* Correspondence: humoora@gmail.com

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Abstract: In this paper, we obtain the boundedness of commutators generated by the Calderón-Zygmund operator, BMO functions and Lipschitz function on Herz-Morrey-Hardy spaces with variable exponent $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$.

Keywords: Simpson's inequality, strongly (s,m)-convex functions, numerical quadrature rule, Hölder inequality.

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1. Introduction

Suppose $\mathbb{S}^{n-1}$, $(n \geq 2)$ denote the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}(\sigma')$. Let Ω be homogeneous function of degree zero and satisfies

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x'), \text{ where } x' = x/|x|(x \neq 0). \tag{1}$$

The Calderón-Zygmund singular integral operator T_Ω is defined as

$$T_\Omega h(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} h(y)\mathrm{d}y. \tag{2}$$

Now, we recall the definitions of the corresponding commutators of the Calderón-Zygmund singular integral operator. Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$, the commutators $[b, T_\Omega]$ generated by b and T is defined as

$$[b, T_\Omega]h(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} [b(x) - b(y)]h(y)\mathrm{d}y. \tag{3}$$

These operators were firstly introduced by Calderón and Zygmund in [1,2], in which they proved that these operator are bounded on L^p, where $0 < p < 1$. Coifman *et al.* [3] showed that if $\Omega \in \dot{\Lambda}_\gamma(\mathbb{S}^{n-1})$ where $\gamma \in (0,1)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, T_\Omega]$ is bounded on L^p. In 2011, Lu Ding and Yan [4] proved that T_Ω and the commutator $[b, T_\Omega]$ are bounded on weighted $(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$.

In last 30 years, the function spaces with variable exponent have attracted researchers since the paper [5] appeared in 1991, see, for example [6–10] and their references. Recently, Jingshi Xu and Xiaodi Yang [11] studied the Herz-Morrey-Hardy spaces with variable exponent and their applications.

Motivated by [11–13], our main purpose of this paper is to study some boundedness for commutators of Calderón−Zygmund operators on Herz-Morrey-Hardy space with two variable exponents. The main tools are properties of variable exponent, BMO function and Lipschitz function.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$ with the Lebesgue measure > 0. For a measurable function $p(\cdot) : \Omega \to [1, \infty)$, the variable Lebesgue space is defined as

$$L^{p(\cdot)}(\Omega) := \left\{h \text{ is measurable on } \Omega : \rho_p(h) < \infty\right\},$$

where
$$\rho_p(h) := \int_\Omega \left(\frac{|h(x)|}{\mu}\right)^{p(x)} dx < \infty \text{ for some constat } \mu > 0.$$
The set $L^{p(\cdot)}(\Omega)$ is a quasi Banach space with following Luxemburg-Nakano norm
$$\|h\|_{L^{p(\cdot)}} := \inf\left\{\mu > 0 : \rho_p(\mu^{-1}h) \le 1\right\}.$$
The space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined as
$$L^{p(\cdot)}_{\text{loc}}(\Omega) := \left\{h : h\chi_k \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for any compact subset } K \subset \Omega\right\}.$$
Suppose $\mathcal{P}(\Omega)$ represents the set of all function $p : \Omega \to [1, \infty)$. Assume that $p_- = \operatorname{ess\,inf}_{x\in\Omega}\ p(x)$ and $p_+ = \operatorname{ess\,sup}_{x\in\Omega}\ p(x)$. Set $p_- > 1$, $p_+ < \infty$ and $p(\cdot)$, $p'(\cdot)$ are conjugate exponent function defined by $1/p(\cdot) + 1/p'(\cdot) = 1$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ satisfying that the maximal function is bounded on $L^{p(\cdot)}$.

Definition 2. (see[11]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q < \infty$, $0 \le \lambda < \infty$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $N > n + 1$. The homogeneous Herz-Morrey-Hardy spaces $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ and nonhomogeneous Herz-Morrey-Hardy spaces $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ are defined as
$$HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n) = \left\{h \in \mathcal{S}'(\mathbb{R}^n) : \|h\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} := \|G_N h\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\right\},$$
$$HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n) = \left\{h \in \mathcal{S}'(\mathbb{R}^n) : \|h\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} := \|G_N h\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\right\}.$$
respectively.

2. Preliminaries and Lemmas

Proposition 3. *(see[14]). Given a function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$. If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies*
$$|p(x) - p(y)| \le \frac{-C}{Log(|x-y|)}; \qquad |x - y| \le 1/2, \tag{4}$$
and
$$|p(x) - p(y)| \le \frac{C}{Log(e+|x|)}; \qquad |y| \ge |x|, \tag{5}$$
then, $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.

Lemma 4. *(see[5]).* (Generalized Hölder Inequality) *Given $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

1. *For every $h \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, we have*
$$\int_{\mathbb{R}^n} |h(x)g(x)|dx \le C\|h\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$
where $C_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.
2. *For every $h \in L^{p_1(\cdot)}(\mathbb{R}^n)$, and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, when $\frac{1}{p(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{p_1(\cdot)}$, we have*
$$\|h(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|g(x)\|_{L^{p_2}(\mathbb{R}^n)}\|h(x)\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},$$
where $C_{p_1,p_2} = [1 + \frac{1}{p_{1-}} - \frac{1}{p_{1+}}]^{\frac{1}{p_-}}$.

Lemma 5. *(see[15,16]). Given $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. If there exist positive constants C, δ_1 and δ_2 such that $\delta_1,\ \delta_2<1$, then for all balls $B\subset\mathbb{R}^n$ and all measurable subset $R\subset B$, we have*

$$\frac{\|\chi_R\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\le C\frac{|R|}{|B|},\quad \frac{\|\chi_R\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\le C\left(\frac{|R|}{|B|}\right)^{\delta_2},\quad \frac{\|\chi_R\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\le C\left(\frac{|R|}{|B|}\right)^{\delta_1}.$$

Lemma 6. *(see[17]). If $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C>0$ such that for any ball B in $\mathbb{R}^n$, we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C.$$

Now, the BMO function and BMO norm are defined as

$$\mathrm{BMO}(\mathbb{R}^n):=\left\{b\in L^1_{loc}(\mathbb{R}^n):\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}<0\right\},$$
$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}:=\sup_{Q:cube}|Q|^{-1}\int_Q|b(x)-b_Q|dx.$$

respectively.

Lemma 7. *(see[8]). Given $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $b\in\mathrm{BMO}(\mathbb{R}^n)$. If $i,j\in\mathbb{Z}$ with $i<j$, then we have*

1. $C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\le\sup_B\frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$.
2. $\|(b-b_{B_i})\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C(j-i)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$.

Lemma 8. *(see[18]). Suppose that $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $q\in[0,\infty)$ and $\lambda\in[0,\infty)$. If $\alpha(\cdot)$ is log-Hölder continuous both at origin and at infinity, then*

$$\|h\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)}\approx max\left\{\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\|h\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q},\right.$$
$$\left.\sup_{L>0,L\in\mathbb{Z}}\left[2^{-L\lambda}\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\|h\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}+2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k\alpha_\infty q}\|h\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\right]\right\}.$$

Lemma 9. *(see[4]). Let Ω satisfies L^r-Dini condition $r\in[1,\infty)$. If there exist constants $C>0$ and $R>0$ such that $|y|<R/2$, then for every $x\in\mathbb{R}^n$, we have*

$$\left(\int_{R<|x|<2R}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\right|^r dx\right)^{\frac{1}{r}}\le CR^{(\frac{n}{r}-n)}\left\{\frac{|y|}{R}+\int_{|y|/2R<\delta<|y|/R}\frac{w_r(\delta)}{\delta}d\delta\right\}.$$

Lemma 10. *(see[11]). Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $q\in[0,\infty)$ and $\lambda\in[0,\infty)$. Let $\alpha(\cdot)$ is log-Hölder continuous both at origin and at infinity. If $2\lambda\le\alpha(\cdot)$, $n\delta_2\le\alpha(0)$, $\alpha<\infty$ and δ_2 as defined in Lemma 5. Then $h\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ $\left(\text{or } HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)\right)$ if and only if $h=\sum_{k=-\infty}^{\infty}\lambda_k g_k$ $\left(\text{or }\sum_{k=0}^{\infty}\lambda_k g_k\right)$, in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each g_k be a central $(\alpha(\cdot),p(\cdot))$-atom (or central $(\alpha(\cdot),\ p(\cdot))$-atom of restricted type) with support contained in B_k and $\sup_{L\in\mathbb{Z}}2^{-L\lambda}\sum_{k=-\infty}^{L}|\lambda_k|^q<\infty$ or $\left(\sup_{L\in\mathbb{Z}}2^{-L\lambda}\sum_{k=0}^{L}|\lambda_k|^q<\infty\right)$.*
Also,

$$\|h\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\approx\inf\sup_{L\in\mathbb{Z}}2^{L\lambda}\left(\sum_{k=-\infty}^{L}|\lambda_k|^q\right)^{1/q}\left(or\ \|h\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\approx\inf\sup_{L\in\mathbb{Z}}2^{L\lambda}\left(\sum_{k=0}^{L}|\lambda_k|^q\right)^{1/q}\right),$$

where infimum is taken over all above decomposition of h.

Lemma 11. *(see [19]). Let $p(\cdot)\in\mathcal{P}(\Omega)$ and $h:\Omega\times\Omega\to\mathbb{R}$ is a measurable function (with respect to product measure) such that, for almost every $y\in\Omega$, $h(\cdot,y)\in L^{p(\cdot)}(\Omega)$. Then*

$$\left\|\int_\Omega h(\cdot,y)dy\right\|_{L^{p(\cdot)}(\Omega)}\le C\int_\Omega\|h(\cdot,y)\|_{L^{p(\cdot)}(\Omega)}dy.$$

Lemma 12. *(see[19]). Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (4) and (5) of Proposition 3, then for any ball (or cube) $Q \subset \mathbb{R}^n$, we have*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}}, & \text{if } |Q| \leq 2^n; \\ |Q|^{\frac{1}{p(\infty)}}, & \text{if } |Q| \geq 1, \end{cases}$$

where $p(\infty) = \lim_{x\to\infty} p(x)$.

3. Main Results

In this section, we formulate and prove the main results of this paper.

Theorem 13. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\Omega \in L^r(\mathbb{S}^{n-1})(r > p^+)$ satisfies*

$$\int_0^1 \frac{w_r(\delta)}{\delta^{1+\beta}} d\delta < \infty, \quad 0 < \beta \leq 1. \tag{6}$$

Suppose that $0 < q < \infty$, $0 \leq \lambda < \infty$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ satisfies conditions (4) and (5) of Proposition 3. If $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0), \alpha_\infty < \beta + n\delta_2$, then T_Ω is bounded from $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ or $\left(HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}\right)$ to $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ or $\left(MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}\right)$.

Proof. It suffices to prove for $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. Assume that $h \in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$, then by Lemma 10, $h = \sum_{j=-\infty}^{\infty} \lambda_j g_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, where $\|h\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \inf \sup_{L\in\mathbb{Z}} 2^{L\lambda}(\sum_{j=-\infty}^{L} |\lambda_j|^q)^{1/q}$, and g_j is a dyadic central $(\alpha(\cdot), p(\cdot))$-atom with support contained in B_j. For simplicity, we take $\Upsilon = \sup_{L\in\mathbb{Z}} 2^{L\lambda} \sum_{j=-\infty}^{L} |\lambda_j|^q$. By virtue of Lemma 8, we have

$$\begin{aligned}
&\|T_\Omega(h)\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)} \approx max\left\{ \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q},\right. \\
&\left.\sup_{L>0, L\in\mathbb{Z}}\left[2^{-L\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} + 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\right]\right\} \\
&\approx max\{E, F+G\}.
\end{aligned}$$

Let

$$\begin{aligned}
E &= \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
F &= \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
G &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{k\alpha_\infty q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

To finish our proof, we only need to show that there exists a constant $C > 0$, such that $E,\ F,\ G \leq C\Upsilon$.
First we prove that $E \leq C\Upsilon$.

$$\begin{aligned}
E &= \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\
&+ \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q := E_1 + E_2.
\end{aligned}$$

By the $\left(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n)\right)$-boundedness of the T_Ω (see[13]), we get

$$\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq |B_j|^{-\alpha_j/n} = 2^{-j\alpha_j}.$$

Therefore, when $0 < q \leq 1$, we obtain

$$\begin{aligned}
E_1 &= \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=k}^{\infty} |\lambda_j| \|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-j\alpha_j} \right)^q \\
&\leq C \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left[\left(\sum_{j=k}^{-1} |\lambda_j| 2^{-j\alpha(0)} + \sum_{j=0}^{\infty} |\lambda_j| 2^{-j\alpha_\infty} \right)^q \right] \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=k}^{-1} |\lambda_j|^q 2^{(k-j)\alpha(0)q} + \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)} \sum_{j=0}^{\infty} |\lambda_j| 2^{-j\alpha_\infty} \right)^q \right] \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j} 2^{(k-j)\alpha(0)q} + \sup_{L\leq 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{(\lambda-\alpha_\infty)jq} 2^{-L\lambda q} \sum_{k=\infty}^{L} 2^{k\alpha(0)q} \right] \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q + \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j} 2^{(k-j)\alpha(0)q} \right. \\
&\qquad \left. + Y \sup_{L\leq 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_\infty)jq} \sum_{k=\infty}^{L} 2^{[k\alpha(0)-L\lambda]q} \right] \\
&\leq C \left[Y + \sup_{L\leq 0, L\in\mathbb{Z}} \sum_{j=L}^{-1} 2^{-j\lambda q} |\lambda_j|^q 2^{(j-L)\lambda q} \sum_{k=-\infty}^{j} 2^{(k-j)\alpha(0)q} + Y \right] \\
&\leq C \left[Y + Y \sup_{L\leq 0, L\in\mathbb{Z}} \sum_{j=L}^{-1} 2^{(j-L)\lambda q} \sum_{k=-\infty}^{j} 2^{(k-j)\alpha(0)q} + Y \right] \leq CY.
\end{aligned}$$

when $1 < q < \infty$, and $1/q + 1/q' = 1$, we have

$$\begin{aligned}
E_1 &= \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=k}^{\infty} |\lambda_j| \|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-j\alpha_j} \right)^q \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=k}^{-1} |\lambda_j| 2^{(k-j)\alpha(0)} \right)^q + \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=0}^{\infty} |\lambda_j| 2^{-j\alpha_\infty} \right)^q \right] \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{(k-j)\alpha(0)\frac{q}{2}} \right) \times \left(\sum_{j=k}^{-1} 2^{(k-j)\alpha(0)\frac{q'}{2}} \right)^{\frac{q}{q'}} \right. \\
&\qquad \left. + \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=0}^{\infty} |\lambda_j|^q 2^{-j\alpha_\infty \frac{q}{2}} \right) \times \left(\sum_{j=0}^{\infty} 2^{-j\alpha_\infty \frac{q'}{2}} \right)^{\frac{q}{q'}} \right] \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=k}^{-1} |\lambda_j|^q 2^{(k-j)\alpha(0)\frac{q}{2}} + \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-j\alpha_\infty \frac{q}{2}} \right] \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=\infty}^{j} 2^{(k-j)\alpha(0)\frac{q}{2}} + \sup_{L\leq 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{(\lambda-\frac{\alpha_\infty}{2})jq} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \right] \\
&\leq C \left[\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q + \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=\infty}^{j} 2^{(k-j)\alpha(0)\frac{q}{2}} \right. \\
&\qquad \left. + Y \sup_{L\leq 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda-\frac{\alpha_\infty}{2})jq} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left[\mathrm{Y}+\sup_{L\leq 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}|\lambda_j|^q 2^{(j-L)\lambda q}\sum_{k=\infty}^{j}2^{(k-j)\alpha(0)\frac{q}{2}}+\mathrm{Y}\right]\\
&\leq C\left[\mathrm{Y}+\mathrm{Y}\sup_{L\leq 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{(j-L)\lambda q}\sum_{k=\infty}^{j}2^{(k-j)\alpha(0)\frac{q}{2}}+\mathrm{Y}\right]\\
&\leq C\mathrm{Y}. \qquad (7)
\end{aligned}
$$

Now, we prove that $E_2 \leq C\mathrm{Y}$. Note that if $x \in A_k$ for each $k \in \mathbb{Z}, y \in A_j$ and $j \leq k-1$. Let $\tilde{p}(\cdot) > 1$ and $1/p(\cdot) = 1/\tilde{p}(\cdot) + 1/r$. Since $r > p^+$, so by Lemma 4 and Lemma 11, we get

$$
\begin{aligned}
\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}|g_j(y)|dy\\
&\leq \int_{B_j}\left\|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right\|_{L^r(\mathbb{R}^n)}\|\chi_k\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n)}|g_j(y)|dy. \qquad (8)
\end{aligned}
$$

Using Lemma 9, we get

$$
\begin{aligned}
\left\|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right\|_{L^r(\mathbb{R}^n)} &\leq C2^{(k-1)(\frac{n}{r}-n)}\left\{\frac{|y|}{2^{k-1}}+\int_{|y|/2^k}^{|y|/2^{k-1}}\frac{w_r(\delta)}{\delta}d\delta\right\}\\
&\leq C2^{(k-1)(\frac{n}{r}-n)}\left(2^{j-k}+2^{(j-k)\beta}\int_0^1\frac{w_r(\delta)}{\delta^{1+\beta}}d\delta\right)\\
&\leq C2^{(k-1)(\frac{n}{r}-n)}2^{(j-k)\beta}. \qquad (9)
\end{aligned}
$$

By Lemma 12, we obtain

$$
\begin{aligned}
\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\right\|_{L^r(\mathbb{R}^n)}\|\chi_k\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n)}\int_{B_j}|g_j(y)|dy\\
&\leq C\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\right\|_{L^r(\mathbb{R}^n)}\left(\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}|B_k|^{\frac{1}{r}}\right)\|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \qquad (10)
\end{aligned}
$$

From (8) and Lemmas 4-6, we get

$$
\begin{aligned}
\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C2^{-nk+(j-k)\beta}\left(\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_k|\right)\|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{(j-k)\beta}\left(\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\right)\|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{(j-k)(\beta+n\delta_2)-j\alpha_j}. \qquad (11)
\end{aligned}
$$

So, when $0 < q \leq 1$, we obtain

$$
\begin{aligned}
E_2 &= \sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&\leq C\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|)2^{(j-k)(\beta+n\delta_2)-j\alpha(0)}\right)^q\\
&\leq C\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]}\right)^q\\
&\leq C\sup_{L\leq 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q\left(\sum_{k=j+1}^{-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]}\right)^q\\
&\leq C\mathrm{Y}. \qquad (12)
\end{aligned}
$$

When $0<q<\infty$, and $1/q+1/q'=1$, by $n\delta_2\le\alpha(0)<\beta+n\delta_2$ and Hölder's inequality, we have

$$\begin{aligned}
E_2 &= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&\le C\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|^q2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}}\right)\\
&\quad\times\left(\sum_{j=-\infty}^{k-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q'}{2}}\right)^{\frac{q}{q'}}\\
&\le C\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q\sum_{k=j+1}^{-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}}\\
&\le CY. \qquad (13)
\end{aligned}$$

Next we prove that $F\le CY$.

$$\begin{aligned}
F &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\le\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&\quad+\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&:= F_1+F_2. \qquad (14)
\end{aligned}$$

Since $0<q\le 1$, we get

$$\begin{aligned}
F_1 &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\le C\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|2^{-j\alpha_j}\right)^q\\
&\le C\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left[\sum_{j=k}^{-1}|\lambda_j|^q2^{-j\alpha(0)q}+\sum_{j=0}^{\infty}|\lambda_j|^q2^{-j\alpha_\infty q}\right]\\
&\le C\left[\sum_{k=-\infty}^{-1}\sum_{j=k}^{-1}|\lambda_j|^q2^{(k-j)\alpha(0)q}+\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\sum_{j=0}^{\infty}|\lambda_j|^q2^{-j\alpha_\infty q}\right]\\
&\le C\left[\sum_{j=-\infty}^{-1}|\lambda_j|^q\sum_{k=-\infty}^{j}2^{(k-j)\alpha(0)q}+\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\sum_{j=0}^{\infty}2^{-j\lambda q}\sum_{l=0}^{j}|\lambda_l|^q2^{(\lambda-\alpha_\infty)jq}\right]\\
&\le C\left[Y+Y\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty)jq}\right]\\
&\le CY. \qquad (15)
\end{aligned}$$

when $0<q<\infty$ and $1/q+1/q'=1$, we deduce

$$\begin{aligned}
F_1 &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\le C\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|2^{-j\alpha_j}\right)^q\\
&\le C\left[\sum_{k=-\infty}^{-1}\left(\sum_{j=k}^{-1}|\lambda_j|2^{(k-j)\alpha(0)}\right)^q+\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=0}^{\infty}|\lambda_j|2^{-j\alpha_\infty}\right)^q\right]\\
&\le C\left[\sum_{k=-\infty}^{-1}\left(\sum_{j=k}^{-1}|\lambda_j|^q2^{(k-j)\alpha(0)\frac{q}{2}}\right)\times\left(\sum_{j=k}^{-1}2^{(k-j)\alpha(0)\frac{q'}{2}}\right)^{\frac{q}{q'}}\right.\\
&\quad\left.+\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=0}^{\infty}|\lambda_j|^q2^{-j\alpha_\infty\frac{q}{2}}\right)\times\left(\sum_{j=0}^{\infty}2^{-j\alpha_\infty\frac{q'}{2}}\right)^{\frac{q}{q'}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left[\sum_{k=-\infty}^{-1}\sum_{j=k}^{-1}|\lambda_j|^q 2^{(k-j)\alpha(0)\frac{q}{2}}+\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\sum_{j=0}^{\infty}|\lambda_j|^q 2^{-j\alpha_\infty\frac{q}{2}}\right]\\
&\leq C\left[\sum_{j=-\infty}^{-1}|\lambda_j|^q\sum_{k=\infty}^{j}2^{(k-j)\alpha(0)\frac{q}{2}}+\sum_{j=0}^{\infty}2^{-j\lambda q}\sum_{l=-\infty}^{j}|\lambda_l|^q 2^{(\lambda-\frac{\alpha_\infty}{2})jq}\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\right]\\
&\leq C\left[\Upsilon\sum_{k=\infty}^{j}2^{(k-j)\alpha(0)\frac{q}{2}}+\Upsilon\sum_{j=0}^{\infty}2^{(\lambda-\frac{\alpha_\infty}{2})jq}\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\right]\\
&\leq C\Upsilon.
\end{aligned}
$$

Now we prove that $F_2\leq C\Upsilon$. When $0<q\leq 1$, from (11) and $n\delta_2\leq\alpha(0)<\beta+n\delta_2$, we have

$$
\begin{aligned}
F_2 &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\leq C\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|2^{(j-k)(\beta+n\delta_2)-j\alpha(0)}\right)^q\\
&\leq C\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]}\right)^q\leq C\sum_{j=-\infty}^{-1}|\lambda_j|^q\left(\sum_{k=j+1}^{-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]}\right)^q\\
&\leq C\Upsilon.
\end{aligned}
$$

when $0<q\leq\infty$, and $1/q+1/q'=1$, by $n\delta_2\leq\alpha(0)<\beta+n\delta_2$ and Hölder's inequality, we obtain

$$
\begin{aligned}
F_2 &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&\leq C\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|^q 2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}}\right)\times\left(\sum_{j=-\infty}^{k-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q'}{2}}\right)^{\frac{q}{q'}}\\
&\leq C\sum_{j=-\infty}^{-1}|\lambda_j|^q\sum_{k=j+1}^{-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}}\\
&\leq C\Upsilon. \qquad (16)
\end{aligned}
$$

Finally we prove that $G\leq C\Upsilon$.

$$
\begin{aligned}
G &= \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{k\alpha_\infty q}\|T_\Omega(h)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\leq\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{k\alpha_\infty q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&+\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{k\alpha_\infty q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&:=G_1+G_2.
\end{aligned}
$$

When $0<q\leq 1$, by the boundedness of the commutator $[b,T]$ in $L^{p(\cdot)}(\mathbb{R}^n)$, we obtain

$$
\begin{aligned}
G_1 &= \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{k\alpha_\infty q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\leq C\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{k\alpha_\infty q}\left(\sum_{j=k}^{\infty}|\lambda_j|2^{-j\alpha_j}\right)^q\\
&\leq C\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{k\alpha_\infty q}\left(\sum_{j=k}^{L}|\lambda_j|^q 2^{-j\alpha_\infty q}+\sum_{j=L}^{\infty}|\lambda_j|^q 2^{-j\alpha_\infty q}\right)\\
&\leq C\left(\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty q}+\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=L}^{\infty}|\lambda_j|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q}\right)\\
&\leq C\left(\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty q}+\sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{j\lambda q-L\lambda q}2^{-j\lambda q}\sum_{l=-\infty}^{j}|\lambda_l|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q}\right)\\
&\leq C\left(\Upsilon+\Upsilon\sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)\lambda q}2^{(L-j)\alpha_\infty q}\right)\leq C\left(\Upsilon+\Upsilon\sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{[(j-L)(\lambda-\alpha_\infty)]q}\right)\leq C\Upsilon. \qquad (17)
\end{aligned}
$$

When $0 < q < \infty$, by using Hölder's inequality, we have

$$\begin{aligned}
G_1 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&\le C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-j\alpha_j\frac{q}{2n}}\right)\times\left(\sum_{j=k}^{\infty}|B_j|^{-j\alpha_j\frac{q'}{2n}}\right)^{\frac{q}{q'}}\\
&\le C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{j=k}^{\infty}|\lambda_j|^q 2^{(k-j)\alpha_\infty\frac{q}{2}}\right)\\
&\le C\left[\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty\frac{q}{2}}+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=L}^{\infty}|\lambda_j|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty\frac{q}{2}}\right]\\
&\le C\left[\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}|\lambda_j|^q+\sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)\lambda q}2^{-j\lambda q}\sum_{l=-\infty}^{j}|\lambda_l|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty\frac{q}{2}}\right]\\
&\le C\left[\Upsilon+\Upsilon\sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)\lambda q}2^{(L-j)\frac{\alpha_\infty}{2}q}\right]\\
&\le C\left[\Upsilon+\Upsilon\sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)(\lambda-\frac{\alpha_\infty}{2})q}\right]\\
&\le C\Upsilon.
\end{aligned}$$

For $G_2 \le C\Upsilon$, when $0 < q < \infty$, and $1/q + 1/q' = 1$, from (11), $n\delta_2 \le \alpha(0)$, $\alpha_\infty < \beta + n\delta_2$ and applying Hölder's inequality, we have

$$\begin{aligned}
G_2 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&\le C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|2^{(j-k)(\beta+n\delta_2)-j\alpha_j}\right)^q\\
&\le C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left(\sum_{j=-\infty}^{-1}|\lambda_j|2^{(j-k)(\beta+n\delta_2)-j\alpha(0)}\right)^q\\
&\quad +C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left(\sum_{j=0}^{k-1}|\lambda_j|2^{(j-k)(\beta+n\delta_2)-j\alpha_\infty}\right)^q\\
&\le C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{[\alpha_\infty-(\beta+n\delta_2)]kq}\left(\sum_{j=-\infty}^{-1}|\lambda_j|2^{[(\beta+n\delta_2)-\alpha(0)]j}\right)^q\\
&\quad +C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{j=0}^{k-1}|\lambda_j|2^{(j-k)[(\beta+n\delta_2)-\alpha_\infty]}\right)^q\\
&\le C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L} 2^{[\alpha_\infty-(\beta+n\delta_2)]kq}\left(\sum_{j=-\infty}^{-1}|\lambda_j|2^{j\frac{[(\beta+n\delta_2)-\alpha(0)]}{2}}\right)^q\times\left(\sum_{j=-\infty}^{-1}(k-j)^{q'}2^{j[(\beta+n\delta_2)-\alpha(0)]\frac{q'}{2}}\right)^{\frac{q}{q'}}\\
&\quad +C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{j=0}^{k-1}|\lambda_j|2^{(j-k)\frac{[(\beta+n\delta_2)-\alpha_\infty]}{2}}\right)^q\times\left(\sum_{j=0}^{k-1}(k-j)^{q'}2^{(j-k)[(\beta+n\delta_2)-\alpha_\infty]\frac{q'}{2}}\right)^{\frac{q}{q'}}\\
&\le C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q 2^{[(\beta+n\delta_2)-\alpha(0)]j\frac{q}{2}}+C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\sum_{j=0}^{k-1}|\lambda_j|^q 2^{(j-k)[(\beta+n\delta_2)-\alpha_\infty]\frac{q}{2}}\\
&\le C\left[\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\sum_{j=0}^{k-1}|\lambda_j|^q 2^{(j-k)[(\beta+n\delta_2)-\alpha_\infty]\frac{q}{2}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left[\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L-1}|\lambda_j|^q\sum_{k=j+1}^{L}2^{(j-k)[(\beta+n\delta_2)-\alpha_\infty]\frac{q}{2}}\right]\\
&\leq C\left[\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L-1}|\lambda_j|^q\right]\\
&\leq C\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L-1}|\lambda_j|^q\leq CY.
\end{aligned}
$$

The proof is completed. □

Theorem 14. *Suppose that $b\in \mathrm{BMO}(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ and let $\Omega\in L^r(\mathbb{S}^{n-1})(r>p^+)$ satisfies (7). Let $0<q<\infty$, $0\leq\lambda<\infty$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ satisfies conditions (4) and (5) of Proposition 3. If $2\lambda\leq\alpha(\cdot)$, $n\delta_2\leq\alpha(0),\alpha_\infty<\beta+n\delta_2$, then $[b,T_\Omega]$ is bounded from $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ or $\left(HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}\right)$ to $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ or $\left(MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}\right)$.*

Proof. It suffices to prove for $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. Set $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $h\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. By Lemma 10, $h=\sum_{j=-\infty}^{\infty}\lambda_j g_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, where $\|h\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\approx\inf\sup_{L\in\mathbb{Z}}2^{L\lambda}\left(\sum_{j=-\infty}^{L}|\lambda_j|^q\right)^{1/q}$, and g_j is a dyadic central $(\alpha(\cdot),p(\cdot))$-atom with support contained in B_j. For simplicity, we denote $Y=\sup_{L\in\mathbb{Z}}2^{L\lambda}\sum_{j=-\infty}^{L}|\lambda_j|^q$. By virtue of Lemma 8, we rewrite

$$
\begin{aligned}
\|T^b_\Omega(h)\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)}&\approx max\left\{\sup_{L\leq0,L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left\|T^b_\Omega(h)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q},\right.\\
&\left.\sup_{L>0,L\in\mathbb{Z}}\left[2^{-L\lambda}\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left\|T^b_\Omega(h)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}+2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k\alpha_\infty q}\left\|T^b_\Omega(h)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\right]\right\}\\
&\approx max\left\{E',F'+G'\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
E'&=\sup_{L\leq0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left\|T^b_\Omega(h)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)},\\
F'&=\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left\|T^b_\Omega(h)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)},\\
G'&=\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{k\alpha_\infty q}\left\|T^b_\Omega(h)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

To complete the prove, we only need to show that there exists a constant $C>0$, such that $E',F',G'\leq CY$. First we show that $E'\leq CY$.

$$
\begin{aligned}
E'&=\sup_{L\leq0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left\|T^b_\Omega(h)\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq\sup_{L\leq0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T^b_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&+\sup_{L\leq0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T^b_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\\
&:=E'_1+E'_2.
\end{aligned}
$$

By the $\left(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n)\right)$-boundedness of the T_Ω^b (see [13]) and following the same way as we estimated E_1 in Theorem 13, we get $E_1' \leq C\|b\|_* \mathrm{Y}$.
Now, we estimate E_2'. For each $k \in \mathbb{Z}$ and $x \in A_k$, by Lemma 7 and Minkowski inequality, we get

$$\begin{aligned}
\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| (b(\cdot) - b(y))\chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} |g_j(y)| \mathrm{d}y \\
&\leq \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b_{B_j}|\chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} |g_j(y)| \mathrm{d}y \\
&\quad + \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} |b_{B_j} - b(y)||g_j(y)| \mathrm{d}y.
\end{aligned}$$

Since $\tilde{p}(\cdot) > 1$ and $1/p(\cdot) = 1/\tilde{p}(\cdot) + 1/r$. Since $r > p^+$, so by Lemma 4 and Lemma 11, we get

$$\begin{aligned}
\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \left\| b(\cdot) - b_{B_j}\chi_k \right\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n)} \int_{B_j} |g_j(y)| dy \\
&\quad + \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n)} \int_{B_j} |b_{B_j} - b(y)||g_j(y)| dy
\end{aligned}$$

From (8) and Lemmas 4-6, we get

$$\begin{aligned}
\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} (k-j)\|b\|_* \left(\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} |B_k|^{\frac{1}{r}} \right) \|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \|(b_{B_j} - b)\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \left(\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} |B_k|^{\frac{1}{r}} \right) \|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} (k-j)\|b\|_* \left(\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} |B_k|^{\frac{1}{r}} \right) \|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \|b\|_* \left(\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} |B_k|^{\frac{1}{r}} \right) \|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_*(k-j)2^{-nk+(j-k)\beta} \left(\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}(\mathbb{R}^n)} |B_k| \right) \|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_*(k-j)2^{(j-k)\beta} \left(\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \right) \|g_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_*(k-j)2^{-j\alpha_j + (j-k)(\beta + n\delta_2)}.
\end{aligned} \tag{18}$$

Therefore, when $0 < q \leq 1$, we obtain

$$\begin{aligned}
E_2' &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C\|b\|_*^q \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|(k-j)2^{(j-k)(\beta+n\delta_2) - j\alpha(0)} \right)^q \\
&\leq C\|b\|_*^q \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|(k-j)2^{[(j-k)(\beta+n\delta_2) - \alpha(0)]} \right)^q \\
&\leq C\|b\|_*^q \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q \left(\sum_{k=j+1}^{-1} (k-j)2^{[(j-k)(\beta+n\delta_2) - \alpha(0)]} \right)^q \\
&\leq C\|b\|_*^q \mathrm{Y}.
\end{aligned}$$

When $0<q\le\infty$, and $1/q+1/q'=1$, by $n\delta_2\le\alpha(0)<\beta+n\delta_2$ and Hölder's inequality, we have

$$\begin{aligned} E_2' &= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\ &\le C\|b\|_*^q \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|^q 2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}}\right) \\ &\times\left(\sum_{j=-\infty}^{k-1}(k-j)^{q'}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q'}{2}}\right)^{\frac{q}{q'}} \\ &\le C\|b\|_*^q \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q\sum_{k=j+1}^{-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}} \\ &\le C\|b\|_*^q Y. \end{aligned}$$

Now we prove that $F'\le CY$.

$$\begin{aligned} F' &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left\|T_\Omega^b(h)\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \le \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=k}^{\infty}|\lambda_j|\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\ &+\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\ &:= F_1'+F_2'. \end{aligned}$$

To estimate F_1'. By the boundedness of the T_Ω^b on $L^{p(\cdot)}(\mathbb{R}^n)$ (see[13]) and following the same way as we estimated F_1 in Theorem 13, we get

$$F_1'\le C\|b\|_*^q Y. \tag{19}$$

For F_2, when $0<q\le 1$, by inequality (18) and $n\delta_2\le\alpha(0)<\varepsilon+n\delta_2$, we have

$$\begin{aligned} F_2' &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\ &\le C\|b\|_*^q\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|(k-j)2^{(j-k)(\beta+n\delta_2)-j\alpha(0)}\right)^q \\ &\le C\|b\|_*^q\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|(k-j)2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]}\right)^q \\ &\le C\|b\|_*^q\sum_{j=-\infty}^{-1}|\lambda_j|^q\left(\sum_{k=j+1}^{-1}(k-j)2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]}\right)^q \\ &\le C\|b\|_*^q Y. \end{aligned} \tag{20}$$

when $0<q\le\infty$, and $1/q+1/q'=1$. by $n\delta_2\le\alpha(0)<\beta+n\delta_2$ and Hölder's inequality, we obtain

$$\begin{aligned} F_2' &= \sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|\|T_\Omega^b(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q \\ &\le C\|b\|_*^q\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|^q 2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}}\right)\times\left(\sum_{j=-\infty}^{k-1}(k-j)^{q'}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q'}{2}}\right)^{\frac{q}{q'}} \\ &\le C\|b\|_*^q\sum_{j=-\infty}^{-1}|\lambda_j|^q\sum_{k=j+1}^{-1}2^{[(j-k)(\beta+n\delta_2)-\alpha(0)]\frac{q}{2}} \\ &\le C\|b\|_*^q Y. \end{aligned}$$

Finally, we show that $G' \leq CY$.

$$
\begin{aligned}
G' &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{k\alpha_\infty q} \left\| T_\Omega^b(h)\chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
&\leq \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{k\alpha_\infty q} \left(\sum_{j=k}^{\infty} |\lambda_j| \| T_\Omega^b(g_j)\chi_k \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{k\alpha_\infty q} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| T_\Omega^b(g_j)\chi_k \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q, \\
&:= G_1' + G_2'.
\end{aligned}
$$

For G_1', by the boundedness of the commutator T_Ω^b in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [13]), and following the same way as we estimated G_1 in Theorem 13, we get

$$
G_1' \leq C\|b\|_*^q Y. \tag{21}
$$

Now, we estimate G_2'. When $0 < q < \infty$, and $1/q + 1/q' = 1$, from inequality (18), since $n\delta_2 \leq \alpha(0)$, $\alpha_\infty < \beta + n\delta_2$ and applying Hölder's inequality, we obtain

$$
\begin{aligned}
G_2' &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{k\alpha_\infty q} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| T_\Omega^b(g_j)\chi_k \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{k\alpha_\infty q} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2) - j\alpha_j} \right)^q \\
&\leq C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{k\alpha_\infty q} \left(\sum_{j=-\infty}^{-1} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2) - j\alpha(0)} \right)^q \\
&\quad + C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{k\alpha_\infty q} \left(\sum_{j=0}^{k-1} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2) - j\alpha_\infty} \right)^q \\
&\leq C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{[\alpha_\infty - (\beta+n\delta_2)]kq} \left(\sum_{j=-\infty}^{-1} |\lambda_j| (k-j) 2^{[(\beta+n\delta_2) - \alpha(0)]j} \right)^q \\
&\quad + C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \left(\sum_{j=0}^{k-1} |\lambda_j| (k-j) 2^{(j-k)[(\beta+n\delta_2) - \alpha_\infty]} \right)^q \\
&\leq C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{[\alpha_\infty - (\beta+n\delta_2)]kq} \left(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{j\frac{[(\beta+n\delta_2) - \alpha(0)]}{2}} \right)^q \left(\sum_{j=-\infty}^{-1} (k-j)^{q'} 2^{j[(\beta+n\delta_2) - \alpha(0)]\frac{q'}{2}} \right)^{\frac{q}{q'}} \\
&\quad + C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{(j-k)\frac{[(\beta+n\delta_2) - \alpha_\infty]}{2}} \right)^q \left(\sum_{j=0}^{k-1} (k-j)^{q'} 2^{(j-k)[(\beta+n\delta_2) - \alpha_\infty]\frac{q'}{2}} \right)^{\frac{q}{q'}} \\
&\leq C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \left(\sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{[(\beta+n\delta_2) - \alpha(0)]j\frac{q}{2}} \right) + C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)[(\beta+n\delta_2) - \alpha_\infty]\frac{q}{2}} \\
&\leq C\|b\|_* \left[\sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)[(\beta+n\delta_2) - \alpha_\infty]\frac{q}{2}} \right] \\
&\leq C\|b\|_* \left[\sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \sum_{k=j+1}^{L} 2^{(j-k)[(\beta+n\delta_2) - \alpha_\infty]\frac{q}{2}} \right] \\
&\leq C\|b\|_* \left[\sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \right] \\
&\leq C\|b\|_* \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L-1} |\lambda_j|^q \\
&\leq C\|b\|_* Y.
\end{aligned}
$$

The proof is completed. $\square$

Theorem 15. *Suppose that* $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)(0 < \gamma \leq 1)$, $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *be such that* $p_1^+ < n/\gamma$, $1/p_1(x) - 1/p_2(x) = \gamma/n$, $\Omega \in L^r(\mathbb{S}^{n-1})(r > q_2^+)$ *with* $1 \leq r' < p_1^-$ *and satisfies*

$$\int_0^1 \frac{w_r(\delta)}{\delta^{1+\gamma}} d\delta < \infty.$$

Let $0 < q < \infty$, $0 \leq \lambda < \infty$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *satisfies conditions (4) and (5) of Proposition 3. If* $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0), \alpha_\infty < \gamma + n\delta_2$, *then* $[b, T_\Omega]$ *is bounded from* $HM\dot{K}^{\alpha(\cdot),q}_{p_1(\cdot),\lambda}$ *or* $\left(HMK^{\alpha(\cdot),q}_{p_1(\cdot),\lambda}\right)$ *to* $M\dot{K}^{\alpha(\cdot),q}_{p_2(\cdot),\lambda}$ *or* $\left(MK^{\alpha(\cdot),q}_{p_2(\cdot),\lambda}\right)$.

Proof. The prove of this Theorem follows almost similarly to that of Theorem 14. Instead of giving all details, we only give the modifications required for the estimation of E'', F'' and G''.

Note that if $x \in B_k$ for each $k \in \mathbb{Z}$, $y \in B_j$ and $j \leq k-1$. Let $\tilde{p}(\cdot) > 1$ and $1/p(\cdot) = 1/\tilde{p}(\cdot) + 1/r$, since $r > p^+$, so by Lemmas 10 and 12, we get

$$\begin{aligned}
\|T^b_\Omega(g_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| (b(\cdot) - b(y))\chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} |g_j(y)| dy \\
&\leq \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b_{B_j}|\chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} |g_j(y)| dy \\
&\quad + \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} |b_{B_j} - b(y)||g_j(y)| dy
\end{aligned}$$

Since $\tilde{p_2}(\cdot) > 1$ and $1/p_2(\cdot) = 1/\tilde{p_2}(\cdot) + 1/r$, by $r > p^+$ and Lemmas 11 and 12, we deduced

$$\begin{aligned}
\|T^b_\Omega(g_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \left\| (b - b_{B_j})\chi_k \right\|_{L^{\tilde{p}_2(\cdot)}(\mathbb{R}^n)} \int_{B_j} |g_j(y)| dy \\
&\quad + \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{p}_2(\cdot)}(\mathbb{R}^n)} \int_{B_j} |b_{B_j} - b(y)||g_j(y)| dy \\
&\leq C \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \left\| (b - b_{B_j})\chi_k \right\|_{L^{\tilde{p}_2(\cdot)}(\mathbb{R}^n)} \|g_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
&\quad + \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right\|_{L^r(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{p}_2(\cdot)}(\mathbb{R}^n)} \|(b - b_{B_j})\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|g_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

By Lemma 9, we have

$$\begin{aligned}
\left\| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right\|_{L^r(\mathbb{R}^n)} &\leq C 2^{(k-1)(\frac{n}{r} - n)} \left\{ \frac{|y|}{2^{k-1}} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{w_r(\delta)}{\delta} d\delta \right\} \\
&\leq C 2^{(k-1)(\frac{n}{r} - n)} \left(2^{j-k} + 2^{(j-k)\gamma} \int_0^1 \frac{w_r(\delta)}{\delta^{1+\gamma}} d\delta \right) \\
&\leq C 2^{(k-1)(\frac{n}{r} - n)} 2^{(j-k)\gamma}. \qquad (22)
\end{aligned}$$

by Lemma 4-6, we have

$$\begin{aligned}
\|T^b_\Omega(g_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq C 2^{(k-1)(\frac{n}{s} - n) + (j-k)\gamma} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} 2^{\gamma k} \left(\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|B_k|^{\frac{1}{r} + \frac{\gamma}{n}}} \right) \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|g_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\quad + C 2^{(k-1)(\frac{n}{s} - n) + (j-k)\gamma} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} 2^{\gamma k} \left(\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{|B_k|^{\frac{1}{r} + \frac{\gamma}{n}}} \right) \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|g_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} 2^{-nk + (j-k)\gamma} \left(\|\chi_{B_k}\|^{-1}_{L^{p_1'(\cdot)}(\mathbb{R}^n)} |B_k| \right) \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|g_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \qquad (23)
\end{aligned}$$

$$\leq C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} 2^{(j-k)\gamma} \left(\frac{\|\chi_j\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}} \right) \|g_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} 2^{(j-k)(\gamma+n\delta_2)-j\alpha_j}. \quad (24)$$

From this, following the same calculations as we did for E', F' and G' in Theorem 14, we get

$$E'', F'', G'' \leq C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \Upsilon. \quad (25)$$

□

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