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Hydromagnetic Stability of Homogeneous Shear Flows in Sea Straits Type Region of Arbitrary Cross Sections

Hari Kishan[1], Nirmal Kumari[1] and Naresh Kumar Dua[2*]

[1]*Department of Mathematics, D.N. College, Meerut (U.P.), India.*
[2]*Department of Mathematics, P.D.M. College of Engineering, Bahadurgarh (Haryana), India.*

Research Article

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Abstract

In this paper, the hydromagnetic stability of homogeneous shear flows in sea straits type region of arbitrary cross section has been discussed. A weak magnetic field is applied parallel to the flow of incompressible fluid. Different bounds of unstable modes have been obtained.

Keywords: Hydromagnetic stability, homogeneous shear flow and sea straits.

1 Introduction

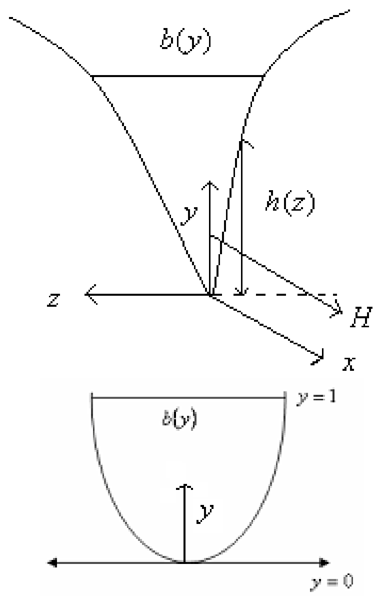
The stability of homogeneous shear flows and stratified shear flows of an inviscid fluid to infinitesimal normal mode disturbances has been studied extensively by several researchers such as [1,2,3,4,5]. These studies are restricted to rectangular cross sections of fluid flows. But in the case of sea straits these cross sections are rarely rectangular. Therefore in the study of such flows it is necessary to consider transversely uniform shear flows contained in straits with arbitrary cross sections. The velocity and stratification are considered to vary in the vertical (y) direction only. [6] Derived an extended version of the Taylor-Goldstein equation for non-rectangular cross section by applying the linear theory. [7] Laid out a more general theory for transversely uniform, time dependent, stratified flow in a channel of arbitrary cross section and established the stability equation of homogeneous shear flows. [8] Proved the boundedness of the wave velocity of neutral modes and the Howard's conjecture for the stability problem of homogeneous shear flows in sea straits of arbitrary cross section. They also obtained two estimates for the growth rates and the two parabolic instability regions.

In this paper, the hydromagnetic stability of homogeneous shear flows in sea straits type region of arbitrary cross section has been discussed. Thus in this paper, the work of [8] has been extended for the case of weak applied magnetic applied along the flow direction.

**Corresponding author: harikishan10@rediffmail.com, ndua_10@yahoo.com;*

2 The Governing Stability Equation

Let the waves be linearly propagating in a shear flow with velocity $U(y)$, magnetic field H applied in x-direction and density ρ_0 (constant). The channel is assumed in x-direction and the bottom elevation $h(z)$ has a single minimum with respect to cross channel coordinate z. The width of the channel at any elevation y is denoted by $b(y)$. If there are several minima of $h(z)$ then $b(y)$ represents the sum of widths of individual topographic troughs. Let u, p and (h_x, h_y, h_z) denote small perturbations from the x velocity, hydrostatic pressure of the background flow and component of perturbations in magnetic field and let w and v denote the associated lateral and vertical velocity components. The shear flow in the strait considered here is shown in the following figure



Cross section of flow

The linearized, inviscid, hydrostatic equations of motion describing these fields are given by

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u+\frac{dU}{dy}v\right]=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_x}{\partial x}, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)w=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+\frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_x}{\partial y} \tag{2}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)v=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_x}{\partial x},\tag{3}$$

$$\frac{\partial h_x}{\partial t}+U\frac{\partial h_x}{\partial x}=H\frac{\partial u}{\partial x}+h_y U',\tag{4}$$

$$\frac{\partial h_y}{\partial t}+U\frac{\partial h_y}{\partial x}=H\frac{\partial v}{\partial x},\tag{5}$$

$$\frac{\partial h_z}{\partial t}+U\frac{\partial h_z}{\partial x}=H\frac{\partial w}{\partial x},\tag{6}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\tag{7}$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0.\tag{8}$$

We have to study the waves for which (h_x, h_y, h_z), *p*, *u*, *v* and *w* are uniform in *y*, implying that the isopycnal surfaces rise and fall uniformly across the channel. Such solutions are dynamically consistent only in the limit of long wave length compared to channel width. Integrating (7) across the channel at any *z* and applying the conditions $w = v\frac{dh}{dy}$ at the two side walls leads to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+T(y)v=0,\tag{9}$$

where $T(y)=b^{-1}\dfrac{db}{dy}=\dfrac{d(\log b)}{dy}$.

Now applying the normal mode technique, i.e. the transformations

$$(u, v, h_x h_y h_z, p)=(\bar{u}(y), \bar{v}(y), \overline{h_x}, \overline{h_y}, \overline{h_z}, \bar{p}(y))e^{i(kx-ct)}$$

in the equations (1) to (9), omitting the bar signs and eliminating all variables except *v*, we get

$$\left[\frac{(bv)'}{b}\right]'-\left[\frac{b(bU')'}{U-c}\right]v+S(v''-k^2v)-k^2v=0.\tag{10}$$

The governing stability equation of homogeneous shear flows in sea straits of arbitrary cross section for the case of weak applied magnetic is obtained by neglecting the term Sv'' in comparison to Sk^2v in the equation (10). Thus we have

$$\left[\frac{(bv)'}{b}\right]' - \left[\frac{b(bU')'}{U-c}\right]v - (S+1)k^2v = 0, \tag{11}$$

The boundary conditions are given by

$$v(0) = 0 = v(D), \tag{12}$$

Here U is the basic velocity $b(y)$ is the width function of the channel, k is the wave number and a prime denotes the differentiation with respect to y, the vertical axis, v is the vertical velocity of the disturbance and S is the magnetic parameter.

3 Stability Analysis

Now we establish the following results:

Theorem 1: The wave velocity of neutral modes is bounded.

Proof: For neutral modes the wave velocity c is real. As shown by Deng et al. for singular neutral modes $U_{\min} \le c \le U_{\max}$ and therefore the wave velocity c is bounded. For non-singular neutral modes $U - c \ne 0$ in flow field (0,D).

Multiplying equation (11) by $b\bar{v}$, integrating the resulting equation over the flow domain (0,D) and using the boundary conditions , we get

$$\int \frac{\left|(bv)'\right|^2}{b} dy + \int k^2 b(S+1)|v|^2 dy + \int \frac{b\left(\frac{U'}{b}\right)}{(U-c)} b|v|^2 = 0. \tag{13}$$

This implies that

$$\int [k^2 b(S+1)(U-c)^2 + b\left(\frac{U'}{b}\right)(U-c)] \frac{b|v|^2}{(U-c)^2} \le 0. \tag{14}$$

This gives the quadratic inequality

$$k^2(S+1)c^2 - \left[2k^2U(S+1) + b\left(\frac{U'}{b}\right)'\right]c + k^2(S+1)U^2 + b\left(\frac{U'}{b}\right)'U \le 0 . \tag{15}$$

The discriminant of inequality (15) is given by $b^2\left(\frac{U'}{b}\right)'^2$ which is greater than or equal to zero.

So the quadratic equation obtained by taking the equality sign in inequality (15) has two real unequal roots. This concludes that *c* satisfying inequality (15) should lie between these two roots. Thus we have

$$U_{\min} + \left[\frac{b\left(\frac{U'}{b}\right)}{2k^2(S+1)}\right]_{\min} - \left[\frac{b\left(\frac{U'}{b}\right)}{2k^2(S+1)}\right]_{\max} \le c \le U_{\min} + 2\left[\frac{b\left(\frac{U'}{b}\right)}{2k^2(S+1)}\right]_{\max} \tag{16}$$

This shows the boundedness of wave velocity of all neutral modes.

In the above analysis we have dropped the term $\int \frac{|bv'|'^2}{b} dy$. By the well known Rayleigh-Ritz inequality, we have

$$\int \frac{|bv'|'^2}{b} dy \ge \frac{\pi^2}{D^2 b_{\max}} \int |bv|^2 dy . \tag{17}$$

Using this inequality in the above analysis we can obtain the bounds for c given by

$$U_{\min} + \left[\frac{b\left(\frac{U'}{b}\right)}{\frac{\pi^2 b_{\min}}{Db_{\max}} + 2k^2(S+1)}\right]_{\min} - \left[\frac{b\left(\frac{U'}{b}\right)}{\frac{\pi^2 b_{\min}}{Db_{\max}} + 2k^2(S+1)}\right]_{\max} \le c$$

$$\le U_{\min} + 2\left[\frac{b\left(\frac{U'}{b}\right)}{\frac{\pi^2 b_{\min}}{Db_{\max}} + 2k^2(S+1)}\right]_{\max} .$$

Theorem 2: The growth rate of an unstable mode is given by

$$k^2 c_i^2 \le \frac{\left| b\left(\frac{U'}{b}\right)' \right| (U_{\max} - U_{\min})}{(S+1)\left[1 + \frac{\pi^2 b_{\min}}{D^2 b_{\max}}\right]}.$$

Proof: The real part of equation (13) is given by

$$\int \frac{\left|(bv)'\right|^2}{b} dy + \int k^2 b(S+1)|v|^2 dy + \int \frac{b\left(\frac{U'}{b}\right)'(U - c_r)}{|U-c|^2} b|v|^2 = 0 \tag{18}$$

Further by the semicircle theorem, we have

$$(U - c_r) \le |U_{\max} - U_{\min}| \text{ and } \frac{1}{|U-c|^2} \le \frac{1}{c_i^2}. \tag{19}$$

By the application of (19) and the Rayleigh-Ritz inequality, equation (18) reduces to

$$\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2 (S+1)\right] \int b|v|^2 dy \le \int \left[\left| b\left(\frac{U'}{b}\right)' \right|_{\max} (U_{\max} - U_{\min}) \right] \frac{b|v|^2}{c_i^2} dy .$$

This gives the estimate of growth rate given by

$$k^2 c_i^2 \le \frac{\left| b\left(\frac{U'}{b}\right)' \right| (U_{\max} - U_{\min})}{(S+1)\left[1 + \frac{\pi^2 b_{\min}}{D^2 b_{\max}}\right]}. \tag{20}$$

From this expression it can be seen that $c_i \to 0$ as $k \to \infty$.

Theorem 3: An estimate for the growth rate of an unstable mode is given by

$$k^2 c_i^2 \le \left(\frac{U_{\max}'^2}{4\left[\left(\frac{\pi^2 b_{\min}}{D^2 b_{\max}} \right) + k^2 \left(S_{\min} + 1 \right) \right]} \right).$$

Proof: For an unstable mode, i.e. $c_i > 0$, $U - c \ne 0$. Therefore $(U-c)^{1/2}$ is well defined. Using the substitution $v = (U-c)^{1/2} G$ in (11), we get

$$\left[\frac{(U-c)(bG)'}{b} \right]' - \left[\frac{U'^2}{4(U-c)} + b\left(\frac{U'}{b} \right)' + (S+1)(U-c)k^2 \right] G = 0. \tag{21}$$

The corresponding boundary conditions are $G(0) = 0 = G(D).$ (22)

Multiplying (21) by $b\overline{G}$ ($\overline{G}$ is the complex conjugate of G), integrating the resulting equation over the flow domain and using the boundary conditions (22), we get

$$\int (U-c)\left[\frac{1}{b}\left| (bG)' \right|^2 + k^2 b(S+1)|G|^2 \right] dy + \int \frac{1}{2}\left(\frac{U'}{b} \right)' |bG|^2 dy + \int \frac{bU'^2}{4(U-c)} |G|^2 dy = 0 \cdot \tag{23}$$

The imaginary part of equation (23) gives

$$\int \left[\frac{1}{b}\left| (bG)' \right|^2 + k^2 b(S+1)|G|^2 \right] dy = \int \frac{bU'^2}{4|U-c|^2} |G|^2 dy. \tag{24}$$

With the help of well known Rayleigh-Ritz inequality, we have

$$\int \frac{\left| (bG)'^2 \right|}{b} dy \ge \left(\frac{\pi^2 b_{\min}}{D^2 b_{\max}} \right) \int b|G|^2 dy. \tag{25}$$

Using the above inequality in (24), we get

$$\left[\left(\frac{\pi^2 b_{\min}}{D^2 b_{\max}} \right) + k^2 \left(S_{\min} + 1 \right) \right] \int b|G|^2 dy \le \frac{bU_{\max}'^2}{4c_i^2} \int b|G|^2 dy.$$

This gives the estimate of growth rate as follows

$$k^2 c_i^2 \le \left(\frac{U'^2_{\max}}{4\left[\left(\frac{\pi^2 b_{\min}}{D^2 b_{\max}}\right) + k^2 (S_{\min} + 1)\right]} \right). \tag{26}$$

This is the required estimate of growth rate of an arbitrary unstable mode.

Theorem 4: If (c, G) is a solution of equation (21) under the condition (22) and $f(y) = b\left(\frac{U'}{b}\right)' - 2U_{\min}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2 (S_{\min} + 1)\right] < 0$ for every y in the flow domain then $c_i^2 \le \lambda\left[c_r + \frac{U_{\max}}{(m-1)}\right]$,

where $\lambda = \left[\frac{U'^2 (m-1)}{2|f(y)|}\right]_{\max}$ and $m = \frac{U_{\max}}{U_{\min}} > 1$.

Proof: The real part of equation (23) gives

$$\int (U - c_r)\left[\frac{1}{b}\left|(bG)'\right|^2 + k^2 b(S+1)|G|^2\right] dy + \int \frac{1}{2}\left(\frac{U'}{b}\right)' |bG|^2 dy + \int \frac{(U - c_r) b U'^2}{4|U - c|^2} |G|^2 dy = 0 \cdot \tag{27}$$

Multiplying equation (24) by $-mc_r$ and adding the resulting equation in (27), we get

$$\int [U - (m+1)c_r]\left[\frac{1}{b}\left|(bG)'\right|^2 + k^2 b(S+1)|G|^2\right] dy + \int \frac{1}{2}\left(\frac{U'}{b}\right)' |bG|^2 dy + \int \frac{(U + (m-1)c_r) b U'^2}{4|U - c|^2} |G|^2 dy = 0 \cdot \tag{28}$$

Now we have

$$[U + (m-1)c_r]_{\min} = U_{\max} > 0 \text{ and } (U - (m+1)c_r)_{\max} = -U_{\min} < 0. \tag{29}$$

Therefore we have

$$\int [U - (m+1)c_r]\left[\frac{1}{b}\left|(bG)'\right|^2 + k^2 b(S+1)|G|^2\right] dy$$

$$\le [U-(m+1)c_r]_{\max}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}}+k^2\left(S_{\min}+1\right)\right]\int b|G|^2\,dy$$

$$\le -U_{\min}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}}+k^2\left(S_{\min}+1\right)\right]\int b|G|^2\,dy\,. \qquad (30)$$

Further from (28), we have

$$\int\left[\frac{1}{2}\left(\frac{U'}{b}\right)' - U_{\min}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}}+k^2\left(S_{\min}+1\right)\right]\right]b|G|^2\,dy$$
$$+\int\frac{(U+(m-1)c_r)bU'^2}{4|U-c|^2}|G|^2\,dy \ge 0\,.$$

This implies that

$$\int[2f(y)c_i^2+U'^2(U+(m-1)c_r)]b|G|^2\,dy\ge 0\,, \qquad (31)$$

where $\quad f(y)=b\left(\frac{U'}{b}\right)' - 2U_{\min}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}}+k^2\left(S_{\min}+1\right)\right]<0\,.$

Therefore there exists a point y_0 in the flow domain such that

$$2f(y_0)c_i^2+U'^2(U(y_0)+(m-1)c_r)>0$$

and hence, we have

$$c_i^2\le\lambda\left[c_r+\frac{U_{\max}}{(m-1)}\right]. \qquad (32)$$

Theorem 5: If (c,G) is a solution of equation (21) under the condition (2.22) and

$g(y)=b\left(\frac{U'}{b}\right)' + 2U_{\max}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}}+k^2\left(S_{\min}+1\right)\right]>0$ for every y in the flow domain then

$c_i^2\le\lambda_1\left[c_r-\frac{U_{\min}}{(m+1)}\right],$

where $\lambda_1 = \left[\dfrac{U'^2(m+1)}{2g(y)}\right]_{\min}$ and $m = \dfrac{U_{\max}}{U_{\min}} > 1$.

Proof: Multiplying equation (24) by mc_r and adding the resulting equation in (27), we get

$$\int [U + (m-1)c_r]\left[\frac{1}{b}\left|(bG)'\right|^2 + k^2 b(S+1)|G|^2\right]dy + \int \frac{1}{2}\left(\frac{U'}{b}\right)' |bG|^2\, dy + \int \frac{(U-(m+1)c_r)bU'^2}{4|U-c|^2}|G|^2\, dy = 0. \tag{33}$$

Therefore we have

$$\int [U + (m-1)c_r]\left[\frac{1}{b}\left|(bG)'\right|^2 + k^2 b(S+1)|G|^2\right]dy \geq [U + (m-1)]_{\min}\int\left[\frac{1}{b}\left|(bG)'\right|^2 + k^2 b(S+1)|G|^2\right] \geq U_{\max}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2(S_{\min}+1)\right]\int b|G|^2\, dy. \tag{34}$$

From equation (34), we have

$$\int\left[\frac{1}{2}\left(\frac{U'}{b}\right)' + U_{\max}\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2(S_{\min}+1)\right]\right] b|G|^2\, dy + \int \frac{(U-(m+1)c_r)bU'^2}{4|U-c|^2}|G|^2\, dy \leq 0. \tag{35}$$

This implies that

$$\int [2g(y)c_i^2 + U'^2(U-(m+1)c_r)]b|G|^2\, dy \leq 0, \tag{36}$$

where $g(y) = b\left(\dfrac{U'}{b}\right)' + 2U_{\max}\left[\dfrac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2(S_{\min}+1)\right] > 0$.

Therefore there exists a point y_1 in the flow domain such that

$$2g(y_1)c_i^2 + U'^2(U(y_1) - (m+1)c_r) < 0.$$

and hence, we have

$$c_i^2 \le \lambda_1\left[c_r - \frac{U_{\min}}{(m+1)}\right]. \quad (37)$$

From the above results, it is concluded that the magnetic parameter S reduces the unstable region. This shows the stabilizing role of the magnetic field.

4 Concluding Remarks

In this paper, the hydromagnetic stability of homogeneous shear flows in sea straits type region of arbitrary cross section has been discussed in the case of weak magnetic field applied in the horizontal direction. Different forms of growth rate of unstable modes have been obtained. The stabilizing role of magnetic field has been shown.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Banerjee MB, Gupta JR, Subbiah M. On Reducing Howard's Semi-circle for Homogeneous Shear Flows. Jour. Math. Anal. Appl. 1988;130(2):398-402.

[2] Banerjee MB, Shandil RG, Vinay Kanwar. A Proof of Howard's Conjecture in Homogeneous Parallel Shear Flow. Proc. Ind. Acad. Sci. 1994; 104:593-595.

[3] Drazin PG, Reid WH. Hydrodynamic Stability. Cambridge; 1981.

[4] Padmini M, Subbiah M. Note on Kuo's Problem. Jour. Math. Anal. Appl. 1993; 173(2); 659-665.

[5] Schmid PJ, Henningson DS. Stabilty and Transition in Shear Flows. Springer; 2001.

[6] Pratt LJ, Deese HE, Murray, Johns W. Continuous Dynamical Modes in Straits having Arbitrary Cross Sections, with Applications to the Bab al Mandab. Jour. Phys. Ocean. 2000; 303:2515-2534.

[7] Deng J, Pratt L, Howard L, Jones C. On Stratified Shear Flow in Sea Straits of Arbitrary Cross Section. Stud. Appl. Math. 2003; 111:409-434.

[8] Subbiah M, Ganesh V. On the Stability of Homogeneous Shear Flows in Sea Straits of Arbitrary Cross Section. Ind. Jour. Pure Appl. Math. 2007;38(1):43-50.

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