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On the Solvability Conditions for the Neumann Boundary Value Problem

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Research Article

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Abstract

In previous work of the first author, a solvability condition of the Neumann boundary value problem for the polyharmonic equation in the unit ball was obtained. This condition has a form of equality to zero of some integral of a linear combination of the boundary functions. In the present paper coefficients of that linear combination are explicitly obtained. In the investigation an arithmetical triangle is arisen. For elements of this triangle a recurrence relation similar to binomial coefficients is derived. It managed to get an explicit solution of the recurrence relation obtained.

Keywords: Neumann boundary value problem; polyharmonic equation; Vandermonde determinant; arithmetical triangle

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1 Introduction

Consider the homogeneous Neumann boundary value problem (see, for example, [(1), (4)]) for the polyharmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^k u(x) = 0,\, x \in S; \quad \frac{\partial^j u}{\partial \nu^j}_{|\partial S} = \varphi_j(s),\, s \in \partial S, \quad j = \overline{1,k}, \tag{1.1}$$

where ν is the outer normal to ∂S. More general boundary value problem containing higher order polynomials on normal derivatives in the boundary condition: $P_j(\partial/\partial\nu)u_{|\partial S} = \varphi_j(s)$ is investigated in (3). Solvability Theorem for this problem was proved. Special case of Theorem from (3) is the following statement.

Let $t^{[k]} = t(t-1)\dots(t-k+1)$ be a factorial power of t. Consider a nonsingular matrix $C_k = \left((-1)^i\binom{j}{i}\right)_{i,j=\overline{0,k-1}}$, where $\binom{j}{i} = 0$ when $j < i$.

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Theorem 1.1. *Solution of the Neumann boundary value problem* (1.1) *for* $\varphi_j \in C^{k-j}(\partial S)$, $j = \overline{1,k}$ *exists then and only then when for any vector* $b_k = (b_k^1, \dots, b_k^k)$ *– a solution of the algebraic equation* $N_k^T(0)b_k = 0$ *the following condition holds*

$$\int_{\partial S} (b_k^1 \varphi_1(x) + \dots + b_k^k \varphi_k(x))\, ds_x = 0,$$

where the matrix $N_k(\lambda)$ *has the form* $N_k(\lambda) = \left((\lambda + 2j - 2)^{[i]}\right)_{i,j=\overline{1,k}} \cdot C_k$.

Investigation of numbers b_k^s, where $s = \overline{1,k}$ and $k \in \mathbb{N}$, normalized in such a way that $b_k^1 > 0$ and $|b_k^k| = 1$ is a subject of present paper. For example, for $k = 1, 2$ it is known (1) that $b_1^1 = 1$ and $b_2^1 = -b_2^2 = 1$. If we compose from the numbers b_k^s an arithmetical triangle similar to the Pascal's triangle (see (2)) and normalize its rows in such a way that $b_k^1 > 0$ and $|b_k^k| = 1$, then it has the form

$$B = \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & -1 & & \\ & b_3^1 & & b_3^2 & & b_3^3 & \\ b_4^1 & & b_4^2 & & b_4^3 & & b_4^4 \\ & & & \dots & & & \end{array}. \tag{1.2}$$

Here rows are indexed by the subscript $k \in \mathbb{N}$ and lines which are parallel to the left side of the triangle B are indexed by the superscript $s \in \mathbb{N}$, $s \le k$.

2 Matrix $N_k(0)$

Let us find solutions of the homogeneous system $N_k^T(0)b_k = 0$ for $k > 1$. For this purpose let us write out the matrix $N_k^T(0)$. It has the form $N_k^T(0) = C_k^T \cdot \left((2i-2)^{[j]}\right)_{i,j=\overline{1,k}}$, where

$$\left((2i-2)^{[j]}\right)_{i,j=\overline{1,k}} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 2^{[1]} & 2^{[2]} & \cdots & 2^{[k]} \\ \vdots & \vdots & \ddots & \vdots \\ (2k-2)^{[1]} & (2k-2)^{[2]} & \cdots & (2k-2)^{[k]} \end{pmatrix}.$$

Because the matrix C_k is a nonsingular one then we arrive to the system of the form

$$\begin{array}{c} 2^{[1]}b_k^1 + \dots + 2^{[k-1]}b_k^{k-1} = -2^{[k]}b_k^k \\ \vdots \\ (2k-2)^{[1]}b_k^1 + \dots + (2k-2)^{[k-1]}b_k^{k-1} = -(2k-2)^{[k]}b_k^k \end{array}. \tag{2.1}$$

Denote a matrix of this linear algebraic system as $A = (A_1, \dots, A_{k-1})$, where $A_i = (2^{[i]}, \dots, (2k-2)^{[i]})^T$, $i = \overline{1, k-1}$ are columns of A and let $A_k = (2^{[k]}, \dots, (2k-2)^{[k]})^T$. Then (2.1) has the form

$$A \begin{pmatrix} b_k^1 \\ \cdots \\ b_k^{k-1} \end{pmatrix} = -b_k^k A_k. \tag{2.2}$$

Denote $\Delta_k^0 = \det A$ and $\Delta_k^s = \det(A_1, \dots, A_{s-1}, A_k, A_{s+1}, \dots, A_{k-1})$, $s = \overline{1, k-1}$, i.e.

$$\Delta_k^s = \begin{vmatrix} 2^{[1]} & \dots & 2^{[s-1]} & 2^{[k]} & 2^{[s+1]} & \dots & 2^{[k-1]} \\ 4^{[1]} & \dots & 4^{[s-1]} & 4^{[k]} & 4^{[s+1]} & \dots & 4^{[k-1]} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (2k-2)^{[1]} & \dots & (2k-2)^{[s-1]} & (2k-2)^{[k]} & (2k-2)^{[s+1]} & \dots & (2k-2)^{[k-1]} \end{vmatrix}.$$

It is not hard to see that

$$\Delta_k^0 = \begin{vmatrix} 2^{[1]} & \dots & 2^{[k-1]} \\ 4^{[1]} & \dots & 4^{[k-1]} \\ \vdots & \ddots & \vdots \\ (2k-2)^{[1]} & \cdots & (2k-2)^{[k-1]} \end{vmatrix} = 2 \cdot 4 \cdots (2k-2) \begin{vmatrix} 1 & \dots & 1^{[k-2]} \\ 1 & \dots & 3^{[k-2]} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & (2k-3)^{[k-2]} \end{vmatrix}$$

$$= (2k-2)!! \begin{vmatrix} 1 & \dots & 1^{k-2} \\ 1 & \dots & 3^{k-2} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & (2k-3)^{k-2} \end{vmatrix} = (2k-2)!!W[1,3,\dots,(2k-3)] = (2k-2)!!(2k-4)!! \cdots 2!!,$$

where $W[\lambda_1, \dots, \lambda_{k-1}]$ denotes the Vandermonde determinant of order $k-1$. Here the equality

$$\begin{vmatrix} \lambda_1^{[0]} & \dots & \lambda_1^{[k-2]} \\ \lambda_2^{[0]} & \dots & \lambda_2^{[k-2]} \\ \vdots & \ddots & \vdots \\ \lambda_{k-1}^{[0]} & \cdots & \lambda_{k-1}^{[k-2]} \end{vmatrix} = \begin{vmatrix} \lambda_1^{0} & \dots & \lambda_1^{k-2} \\ \lambda_2^{0} & \dots & \lambda_2^{k-2} \\ \vdots & \ddots & \vdots \\ \lambda_{k-1}^{0} & \cdots & \lambda_{k-1}^{k-2} \end{vmatrix} = W[\lambda_1, \dots, \lambda_{k-1}]$$

was used. Therefore the system (2.1) is uniquely solvable for any b_k^k. Choose $b_k^k = \pm 1$, but so that $b_k^1 > 0$.

To calculate Δ_k^s consider more general case. Let $\lambda_1, \dots, \lambda_{k-1} \in \mathbb{C}$ and $k > 1$. Denote

$$\Delta_k^s(\lambda) \equiv \Delta_k^s(\lambda_1, \dots, \lambda_{k-1}) = \begin{vmatrix} \lambda_1^{[1]} & \dots & \lambda_{k-1}^{[1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[s-1]} & \cdots & \lambda_{k-1}^{[s-1]} \\ \lambda_1^{[k]} & \cdots & \lambda_{k-1}^{[k]} \\ \lambda_1^{[s+1]} & \cdots & \lambda_{k-1}^{[s+1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[k-1]} & \cdots & \lambda_{k-1}^{[k-1]} \end{vmatrix} = (-1)^{k-s-1} \begin{vmatrix} \lambda_1^{[1]} & \dots & \lambda_{k-1}^{[1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[s-1]} & \cdots & \lambda_{k-1}^{[s-1]} \\ \lambda_1^{[s+1]} & \cdots & \lambda_{k-1}^{[s+1]} \\ \vdots & \ddots & \vdots \\ \lambda_1^{[k-1]} & \cdots & \lambda_{k-1}^{[k-1]} \\ \lambda_1^{[k]} & \cdots & \lambda_{k-1}^{[k]} \end{vmatrix} \tag{2.3}$$

and $\Delta_k^0(\lambda) = \det \left(\lambda_j^{[i]} \right)_{i,j=\overline{1,k-1}}$. Dimension of all $\Delta_k^s(\lambda)$ $(s = \overline{0, k-1})$ is $k-1$.

Lemma 2.1. *For integer $k > 1$ and any integer s such that $1 \le s \le k$ the following equality holds*

$$s^{[1]}\Delta_k^1(\lambda) + \dots + s^{[s]}\Delta_k^s(\lambda) = -\Delta_k^0(\lambda) s \prod_{i=1}^{k-1} (s - \lambda_i). \tag{2.4}$$

Proof. Let s be an integer and such that $1 \le s \le k-1$. Consider determinant of the form

$$V(s;\lambda) = \begin{vmatrix} s^{[1]} & \lambda_1^{[1]} & \dots & \lambda_{k-1}^{[1]} \\ \vdots & & \ddots & \vdots \\ s^{[s-1]} & \lambda_1^{[s-1]} & \dots & \lambda_{k-1}^{[s-1]} \\ s^{[s]} & \lambda_1^{[s]} & \dots & \lambda_{k-1}^{[s]} \\ 0 & \lambda_1^{[s+1]} & \dots & \lambda_{k-1}^{[s+1]} \\ \vdots & & \ddots & \vdots \\ 0 & \lambda_1^{[k]} & \cdots & \lambda_{k-1}^{[k]} \end{vmatrix},$$

having dimension k. Open it by the first column. According to (2.3) and taking into account the equality $s^{[i]} = 0$, which is true for $i > s$, we have

$$V(s;\lambda) = (-1)^{k-2} s^{[1]} \Delta_k^1(\lambda) - (-1)^{k-3} s^{[2]} \Delta_k^2(\lambda) + \cdots + (-1)^{s-1}(-1)^{k-s-1} s^{[s]} \Delta_k^s(\lambda) = (-1)^{k-2}\left(s^{[1]}\Delta_k^1(\lambda) + \cdots + s^{[s]}\Delta_k^s(\lambda)\right).$$

On the other hand. Take out the multiplier s from the first column, and the multiplier λ_{i-1} from the ith column for $i > 2$. It is not hard to see that the obtained determinant with the help of linear combination of rows lead to the Vandermonde determinant $W[s, \lambda_1, \ldots, \lambda_{k-1}]$. Therefore

$$V(s;\lambda) = s\lambda_1 \ldots \lambda_{k-1} W[s, \lambda_1, \ldots, \lambda_{k-1}] = s\prod_{i=1}^{k-1} \lambda_i(\lambda_i - s) W[\lambda_1, \ldots, \lambda_{k-1}] = s\prod_{i=1}^{k-1}(\lambda_i - s)\Delta_k^0(\lambda) = (-1)^{k-1} s \prod_{i=1}^{k-1}(s-\lambda_i)\Delta_k^0(\lambda).$$

So, we obtain the equality

$$(-1)^{k-1} s \prod_{i=1}^{k-1}(s-\lambda_i)\Delta_k^0(\lambda) = (-1)^{k-2}\left(s^{[1]}\Delta_k^1(\lambda) + \cdots + s^{[s]}\Delta_k^s(\lambda)\right).$$

From here it follows (2.4). □

3 Auxiliary Identities for Numbers b_k^s

Let us use Lemma 2.1 to find the numbers b_k^s.

Lemma 3.1. *For every integer* $k > 1$ *and* s *such that* $1 \le s \le k-1$ *the following equality holds*

$$s^{[1]} b_k^1 + \cdots + s^{[s]} b_k^s = (-1)^{k+1} 2^k \left(\frac{s}{2}\right)^{[k]}. \tag{3.1}$$

Proof. According to the above notation $\Delta_k^s(2, \ldots, 2k-2) = \Delta_k^s$ and therefore by Lemma 2.1 for $\lambda_i = 2i$ we have

$$s^{[1]}\Delta_k^1 + \cdots + s^{[s]}\Delta_k^s = -\Delta_k^0 \prod_{i=0}^{k-1}(s-2i) = -2^k \left(\frac{s}{2}\right)^{[k]} \Delta_k^0. \tag{3.2}$$

Hence for $s = 1$ and $k > 1$ we obtain

$$\Delta_k^1 = -\Delta_k^0 2^k \left(\frac{1}{2}\right)^{[k]} = (-1)^k \Delta_k^0 (2k-3)!!.$$

Remembering system (2.2) by the Cramer's rule we find

$$b_k^s = \frac{\det(A_1, \ldots, A_{s-1}, -b_k^k A_k, A_{s+1}, \ldots, A_{k-1})}{\det A} = -b_k^k \frac{\Delta_k^s}{\Delta_k^0}.$$

For $s = 1$ we have

$$b_k^1 = -b_k^k \frac{\Delta_k^1}{\Delta_k^0} = (-1)^{k+1} b_k^k (2k-3)!!$$

and hence $b_k^1 > 0$ if $b_k^k = (-1)^{k+1}$. Thus $b_k^s = (-1)^k \dfrac{\Delta_k^s}{\Delta_k^0}$. Therefore, dividing the equality (3.2) by $\Delta_k^0 \neq 0$ and multiplying it by $(-1)^k$ we obtain (3.1). □

From the proof of Lemma 3.1 it follows that

$$b_k^1 = (2k-3)!! \text{ for } k > 1; \quad b_k^k = (-1)^{k+1} \text{ for } k \geq 1. \tag{3.3}$$

Hence, we have found the left and right sides of the arithmetical triangle B, consisting of numbers b_k^s (1.2).

Lemma 3.2. *For $k \in \mathbb{N}$ and integer s such that $1 \leq s \leq k-1$ the numbers b_k^s are defined by the equality*

$$b_k^s = (-1)^{k+s} \frac{2^k}{s!} \sum_{j=1}^{s} (-1)^{j+1} \binom{s}{j} \left(\frac{j}{2}\right)^{[k]}. \tag{3.4}$$

Proof. Let us define $\tilde{b}_k^s = s! b_k^s$ and suppose that $\binom{a}{b} = 0$ for $b > a$. Then (3.1) can be written in the form

$$\binom{s}{1}\tilde{b}_k^1 + \cdots + \binom{s}{s}\tilde{b}_k^s + \cdots + \binom{s}{k-1}\tilde{b}_k^{k-1} = (-1)^{k+1} 2^k \left(\frac{s}{2}\right)^{[k]}$$

and therefore the vector $\tilde{b}_k = (\tilde{b}_k^1, \ldots, \tilde{b}_k^{k-1})$ is a solution of the algebraic system

$$C' \tilde{b}_k = (-1)^{k+1} 2^k \left((\frac{j}{2})^{[k]} \right)_{j=\overline{1,k-1}}$$

with the matrix $C' = \left(\binom{i}{j}\right)_{i,j=\overline{1,k-1}}$. It is nor hard to check that the triangular matrix C' is inversible and $C'^{-1} = \left((-1)^{i+j}\binom{i}{j}\right)_{i,j=\overline{1,k-1}}$. This follows from the following equality ($i \geq j$)

$$\sum_{s=1}^{k-1} (-1)^{s+j} \binom{i}{s}\binom{s}{j} = (-1)^j \sum_{s=j}^{i} (-1)^s \binom{i}{s}\binom{s}{j} = (-1)^j \frac{i!}{j!} \sum_{k=0}^{i-j} \frac{(-1)^{k+j}}{(i-j-k)!k!}$$

$$= \frac{i!}{(i-j)!j!} \sum_{k=0}^{i-j} (-1)^k \frac{(i-j)!}{(i-j-k)!k!} = \binom{i}{j} \delta_{i-j,0} = \delta_{i,j}.$$

Therefore

$$\tilde{b}_k = (-1)^{k+1} 2^k \left(\sum_{j=1}^{k-1} (-1)^{i+j} \binom{i}{j} \left(\frac{j}{2}\right)^{[k]} \right)_{i=\overline{1,k-1}}$$

and then using that $\tilde{b}_k^s = s! b_k^s$ we obtain

$$b_k^s = (-1)^{k+s} \frac{2^k}{s!} \sum_{j=1}^{k-1} (-1)^{j+1} \binom{s}{j} \left(\frac{j}{2}\right)^{[k]}.$$

From here it follows (3.4). □

Let us check the formula (3.4) for $s = 1$. We have $b_k^1 = (-1)^{k+1} 2^k \left(\frac{1}{2}\right)^{[k]} = (2k-3)!!$, which coincides with the obtained above. Calculate the unknown number b_k^2. Form (3.4) for $k > 1$ we obtain

$$b_k^2 = (-1)^{k+2} 2^k \left(\frac{1}{2}\right)^{[k]} = -b_k^2 = -(2k-3)!!. \tag{3.5}$$

Thus we have found the values on the first cross row witch is parallel to the left side of the triangle (1.2). For $k = 2$ we have the known value of $b_2^2 = -1$. Let us try to transform the formula (3.4) to more compact form. Denote $f^{(k)}(t) = \left(\frac{d}{dt}\right)^k f(t)$.

Lemma 3.3. *For $k \in \mathbb{N}$ and integer s such that $1 \leq s \leq k$ the following equality holds*

$$b_k^s = (-1)^{k+1} \frac{2^k}{s!} \left((\sqrt{t} - 1)^s \right)^{(k)}_{|t=1}. \tag{3.6}$$

Proof. It is not hard to see that

$$\left(\frac{j}{2}\right)^{[k]} = \frac{j}{2}\left(\frac{j}{2} - 1\right) \dots \left(\frac{j}{2} - k + 1\right) = \left(t^{\frac{j}{2}}\right)^{(k)}_{|t=1}.$$

Therefore for integer $k > 1$ and s such that $1 \leq s \leq k-1$, according to (3.4), the following equalities take place

$$b_k^s = (-1)^{k+s} \frac{2^k}{s!} \sum_{j=1}^{s} (-1)^{j+1} \binom{s}{j} \left(\frac{j}{2}\right)^{[k]} = (-1)^{k+1} \frac{2^k}{s!} \sum_{j=1}^{s} (-1)^{s-j} \binom{s}{j} \left((\sqrt{t})^j\right)^{(k)}_{|t=1}$$

$$= (-1)^{k+1} \frac{2^k}{s!} \left(\sum_{j=1}^{s} (-1)^{s-j} \binom{s}{j} (\sqrt{t})^j \right)^{(k)}_{|t=1} = (-1)^{k+1} \frac{2^k}{s!} \left((\sqrt{t} - 1)^s \right)^{(k)}_{|t=1},$$

i.e. (3.6) it true for the mentioned k and s. Formula (3.6) is correct for $s = k \geq 1$ too. Indeed, substituting $s = k$ to (3.6) we obtain

$$b_k^k = (-1)^{k+1} \frac{2^k}{k!} \left((\sqrt{t} - 1)^k \right)^{(k)}_{|t=1} = (-1)^{k+1} \frac{2^k}{k!} \left(\frac{k}{2\sqrt{t}} (\sqrt{t} - 1)^{k-1} \right)^{(k-1)}_{|t=1}$$

$$= \frac{(-1)^{k+1}}{\sqrt{t}} \frac{2^{k-1}}{(k-1)!} \left((\sqrt{t} - 1)^{k-1} \right)^{(k-1)}_{|t=1} = -b_{k-1}^{k-1} = (-1)^{k-1} b_1^1.$$

If substitute $s = k = 1$ to (3.6), then we have $b_1^1 = 1$ and therefore $b_k^k = (-1)^{k-1} = (-1)^{k+1}$. Earlier was obtained that $b_k^k = (-1)^{k+1}$ and hence the formula (3.6) is valid for $s = k$. □

Let us make one more transformation to formula (3.6) to calculate the numbers b_k^s. We prove one auxiliary Lemma.

Lemma 3.4. *Let $f \in C^k(a,b)$, then for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$*

$$(f^s(t))^{(k)} = \sum_{i_1 + \ldots + i_s = k} \binom{k}{i_1 \ldots i_s} f^{(i_1)}(t) \cdots f^{(i_s)}(t), \quad t \in (a,b). \tag{3.7}$$

Proof. For $k = 0$ this formula is obvious. For $k = 1$ we have the equality $(f^s(t))' = s f^{s-1}(t) f'(t)$ and by the formula (3.7) we obtain

$$\sum_{i_1 + \ldots + i_s = 1} \binom{1}{i_1 \ldots i_s} f^{(i_1)}(t) \cdots f^{(i_s)}(t) = f'(t) \left(f^{(0)}(t)\right)^{s-1} +$$

$$\cdots + \left(f^{(0)}(t)\right)^{s-1} f'(t) = s f^{s-1}(t) f'(t).$$

This means that (3.7) is true for $k = 1$. Now let (3.7) be true for some $k \in \mathbb{N}$. Prove it for $k + 1$.

Indeed, using the assumption we can write

$$(f^s(t))^{(k+1)} = \left((f^s(t))^{(k)}\right)' = \left(\sum_{i_1+\dots+i_s=k} \binom{k}{i_1 \dots i_s} f^{(i_1)}(t) \cdots f^{(i_s)}(t)\right)' =$$
$$= \sum_{i_1+\dots+i_s=k} \binom{k}{i_1 \dots i_s} \left(f^{(i_1+1)}(t) \cdots f^{(i_s)}(t) + \dots + f^{(i_1)}(t) \cdots f^{(i_s+1)}(t)\right) =$$
$$= \sum_{i_1+\dots+i_s-1=k} \left(\binom{k}{i_1-1 \dots i_s} + \dots + \binom{k}{i_1 \dots i_s-1}\right) f^{(i_1)}(t) \cdots f^{(i_s)}(t) =$$
$$= \sum_{i_1+\dots+i_s=k+1} \binom{k+1}{i_1 \dots i_s} f^{(i_1)}(t) \cdots f^{(i_s)}(t).$$

This is what we desired. By the induction method Lemma is proved. □

4 Recurrence Relations

For $k, s \in \mathbb{N}$ consider the notation

$$a_k^s = \left((\sqrt{t}-1)^s\right)^{(k)}_{|t=1}. \tag{4.1}$$

Investigate a triangle A, which is similar to the triangle (1.2), but consisting from the numbers a_k^s for $k, s \in \mathbb{N}$ and $s \leq k$. If $s = 0$ or $k = 0$, then the values of a_k^s can be calculated as $a_k^0 = a_0^s = 0$. Besides by (4.1) $a_1^1 = \frac{1}{2}$ and $a_0^0 = 1$.

Lemma 4.1. *For the numbers a_k^s the recurrence relation*

$$a_{k+1}^s = \left(\frac{s}{2} - k\right) a_k^s + \frac{s}{2} a_k^{s-1}, \tag{4.2}$$

holds, $a_k^s = 0$ for $s > k$ and $a_k^0 = 0$ for $k \in \mathbb{N}$, but $a_0^0 = 1$.

Proof. It is not hard to notice that if the degree s is more then the order of derivative k, then from (4.1) it follows that $a_k^s = 0$. The values of a_1^1 and a_0^0 are calculated before this Lemma by the formula (4.1). Let us use (3.7) from Lemma 3.4 provided $f(t) = \sqrt{t} - 1$. Because

$$(\sqrt{t}-1)^{(m)} = \frac{1}{2}\left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-m+1\right) t^{\frac{1}{2}-m} = \frac{(-1)^{m-1}}{2^m}(2m-3)!!\, t^{\frac{1}{2}-m},$$

we can write

$$\left((\sqrt{t}-1)^s\right)^{(k)} = \sum_{i_1+\dots+i_s=k,\, i_j>0} \binom{k}{i_1 \dots i_s} (-1)^{k-s} \frac{(2i_1-3)!! \cdots (2i_s-3)!!}{2^k} t^{\frac{s}{2}-k}$$
$$+ s \sum_{i_1+\dots+i_{s-1}=k,\, i_j>0} \binom{k}{i_1 \dots i_{s-1}} (-1)^{k-s+1} \frac{(2i_1-3)!! \cdots (2i_{s-1}-3)!!}{2^k} t^{\frac{s-1}{2}-k}(\sqrt{t}-1)$$
$$+ s(s-1) \sum_{i_1+\dots+i_{s-2}=k,\, i_j>0} \binom{k}{i_1 \dots i_{s-2}} (-1)^{k-s+2} \frac{(2i_1-3)!! \cdots (2i_{s-2}-3)!!}{2^k} t^{\frac{s-2}{2}-k}(\sqrt{t}-1)^2$$
$$+ \dots = a_k^s t^{\frac{s}{2}-k} + s a_k^{s-1} t^{\frac{s-1}{2}-k}(\sqrt{t}-1) + s(s-1) a_k^{s-2} t^{\frac{s-2}{2}-k}(\sqrt{t}-1)^2 + \cdots. \tag{4.3}$$

Therefore

$$\left((\sqrt{t}-1)^s\right)^{(k+1)} = a^s_{k+1}t^{\frac{s}{2}-k-1} + sa^{s-1}_{k+1}t^{\frac{s-1}{2}-k-1}(\sqrt{t}-1) + \cdots .$$

On the other hand if differentiate (4.3) then we obtain

$$\left((\sqrt{t}-1)^s\right)^{(k+1)} = \left(\frac{s}{2}-k\right)a^s_k t^{\frac{s}{2}-k-1} + \frac{s}{2}a^{s-1}_k t^{\frac{s-2}{2}-k} + (\sqrt{t}-1)F(t).$$

Assuming in the previous equalities $t=1$ we obtain

$$a^s_{k+1} = \left(\frac{s}{2}-k\right)a^s_k + \frac{s}{2}a^{s-1}_k,$$

which was to be proved. □

Notice that an equation, similar to (4.2) $a^s_{k+1} = (k-2s+1)a^s_k + \frac{1}{2}a^{s-1}_k$ was used in (5) when constructing a system of special polynomials.

Values of a^s_k with the help of (4.2) were calculated for several cross rows of the triangle A, which are parallel to the sides of the triangle, and according to the obtained formulas we made an assumption about the values of a^s_k. Omitting these calculations we just prove the found formula.

Lemma 4.2. *Solution of the recurrence relation* (4.2) *under the boundary conditions* $a^0_k = a^{k+1}_k = 0$*,* $k \in \mathbb{N}$ *and* $a^0_0 = 1$ *is unique and has the form*

$$a^s_k = (-1)^{k+s}\frac{(2k-s-1)!s}{2^k(2k-2s)!!} = (-1)^{k+s}\frac{(2k-s-1)!s}{2^{2k-s}(k-s)!}, \tag{4.4}$$

where $s, k \in \mathbb{N}$ *and* $s \le k$.

Proof. Prove the existence and uniqueness of the solution of equation (4.2) under the mentioned boundary conditions. Sketch the problem

$$\begin{array}{c} 1 \quad 0 \\ 0 \quad a^1_1 \quad 0 \\ 0 \quad a^1_2 \quad a^1_2 \quad 0 \\ 0 \quad a^1_3 \quad a^2_3 \quad a^3_3 \quad 0 \\ \cdots a^s_{k+1} = (s/2-k)a^s_k + s/2a^{s-1}_k \cdots \end{array}$$

1^0. Assuming $k=0$ and $s=1$ in (4.2), according to the boundary conditions, we obtain $a^1_1 = \frac{1}{2}a^1_0 + \frac{1}{2}a^0_0 = \frac{1}{2}$. Formula (4.4) gives the same value for $k=s=1$.

2^0. Calculate the values of a^1_k using the formula (4.2). Taking into consideration that $a^0_k = 0$ we find

$$a^1_k = \left(\frac{1}{2}-k+1\right)a^1_{k-1} + \frac{1}{2}a^0_{k-1} = -\frac{2k-3}{2}a^1_{k-1} = (-1)^{k-1}\frac{(2k-3)!!}{2^{k-1}}a^1_1 =$$
$$= (-1)^{k-1}\frac{(2k-3)!!}{2^k} = (-1)^{k+1}\frac{(2k-1)!!}{2^k(2k-1)}.$$

This formula was derived under the condition $k>1$, but it is true for $k=1$ too. By the formula (4.4) we obtain the same value

$$a^1_k = (-1)^{k+1}\frac{(2k-2)!}{2^k(2k-2)!!} = (-1)^{k+1}\frac{(2k-3)!!}{2^k}.$$

3^0. Calculate the values of a^k_k using the formula (4.2). Taking into consideration that $a^{k+1}_k = 0$ we find

$$a^k_k = \left(\frac{k}{2}-k+1\right)a^k_{k-1} + \frac{k}{2}a^{k-1}_{k-1} = \frac{k}{2}a^{k-1}_{k-1} = \frac{k!}{2^{k-1}}a^1_1 = \frac{k!}{2^k}.$$

This formula was derived under the condition $k > 1$, but using the equality $a_1^1 = \frac{1}{2}$ we conclude that it is true for $k \geq 1$ too. By the formula (4.4) we obtain the same value

$$a_k^k = \frac{(2k-k-1)!k}{2^k(2k-2k)!!} = \frac{k!}{2^k}.$$

Thus the values of a_k^s on the left and right sides of the triangle A are obtained and the formula (4.4) gives a solution of the equation (4.2) on the both sides of the triangle A. Equation (4.2) has such a structure that by the known values of a_k^s on the both sides of the triangle A all values of a_k^s inside the triangle A are uniquely determined from the equation (4.2). Therefore if numbers a_k^s, defined from (4.4), are satisfied to equation (4.2) inside the triangle A, then formula (4.4) is determined a solution of equation (4.2). Let us prove it. Let a_k^s be found from (4.4). Substitute them to the right-hand side of (4.2). We have

$$a_{k+1}^s = \left(\frac{s}{2} - k\right) a_k^s + \frac{s}{2} a_k^{s-1} = (-1)^{k+s} \left(\frac{s}{2} - k\right) \frac{(2k-s-1)!s}{2^k(2k-2s)!!} + (-1)^{k+s-1} \frac{s}{2} \frac{(2k-s)!(s-1)}{2^k(2k-2s+2)!!}$$
$$= (-1)^{k+s-1} \frac{(2k-s)!s}{2^{k+1}(2k-2s+2)!!} ((2k-2s+2)+(s-1)) = (-1)^{k+s+1} \frac{(2k-s+1)!s}{2^{k+1}(2k-2s+2)!!},$$

i.e. a_{k+1}^s can be found from (4.4) too. Proof is complete. □

5 Triangle B and the Neumann Boundary Value Problem

Let us come back to the triangle B.

Lemma 5.1. *Numbers b_k^s, $s, k \in \mathbb{N}$ are the unique solution of the recurrence relation*

$$b_{k+1}^s = (2k-s)b_k^s - b_k^{s-1} \tag{5.1}$$

under the boundary conditions $b_k^1 = \dfrac{(2k-1)!!}{(2k-1)}$ and $b_k^k = (-1)^{k+1}$, $k \in \mathbb{N}$ (according to (3.3)*) and are defined by the equality*

$$b_k^s = (-1)^{s+1} \binom{2k-s-1}{s-1} \frac{(2k-2s+1)!!}{(2k-2s+1)}. \tag{5.2}$$

Proof. By the formulas (3.6), (4.1) and (4.4) we write

$$b_k^s = (-1)^{k+1} \frac{2^k}{s!} a_k^s = (-1)^{s+1} \frac{(2k-s-1)!}{(2k-2s)!!(s-1)!} = (-1)^{s+1} \binom{2k-s-1}{s-1} \frac{(2k-2s+1)!!}{(2k-2s+1)},$$

i.e. formula (5.2) defines the triangle B. On the both sides of the triangle B formula (5.2) gives

$$b_k^1 = \binom{2k-2}{0} \frac{(2k-1)!!}{(2k-1)} = (2k-3)!!, \quad b_k^k = (-1)^{k+1} \binom{2k-k-1}{k-1} \frac{(2k-2k+1)!!}{(2k-2k+1)} = (-1)^{k+1},$$

which coincide with the boundary conditions. If substitute the value of $a_k^s = (-1)^{k+1} \dfrac{s!}{2^k} b_k^s$ to the equation (4.2), then we obtain

$$(-1)^{k+2} \frac{s!}{2^{k+1}} b_{k+1}^s = \left(\frac{s}{2} - k\right) (-1)^{k+1} \frac{s!}{2^k} b_k^s + \frac{s}{2} (-1)^{k+1} \frac{(s-1)!}{2^k} b_k^{s-1}.$$

After canceling on $(-1)^{k+2} \dfrac{s!}{2^{k+1}}$ we obtain the equation (5.1)

$$b_{k+1}^s = (2k-s)b_k^s - b_k^{s-1}.$$

Notice that for the binomial coefficients we have the relation $C_{k+1}^s = C_k^s + C_k^{s-1}$. □

Equation of the form (5.1) is determined the arithmetical triangle similar to the Pascal's triangle, Eulerian and Stirling numbers. Such triangles are investigated by the modulo of a prime (see, for example, [(6), (2)]).

Let us calculate the values of b_k^s for some first cross rows that are parallel to the both sides of the triangle B. We have

$$b_k^2 = -\binom{2k-3}{1}\frac{(2k-3)!!}{(2k-3)} = -(2k-3)!! = -b_k^1, \quad k \geq 2,$$

that corresponds to (3.5),

$$b_k^{k-1} = (-1)^k\binom{k}{k-2}\frac{3!!}{3} = (-1)^k\binom{k}{2}, \quad k \geq 2$$

and

$$b_k^{k-2} = (-1)^{k-1}\binom{k+1}{k-3}\frac{5!!}{5} = (-1)^{k-1}3\binom{k+1}{4}, \quad k \geq 3.$$

Calculating b_k^s by the formula (5.2) we extend the triangle B from (1.2) up to six rows

$$B = \begin{array}{c} 1 \\ 1 \quad -1 \\ 3 \quad -3 \quad 1 \\ 15 \quad -15 \quad 6 \quad -1 \\ 105 \quad -105 \quad 45 \quad -10 \quad 1 \\ 945 \quad -945 \quad 420 \quad -105 \quad 15 \quad -1 \\ \cdots \quad b_{k+1}^s = (2k-s)b_k^s - b_k^{s-1} \quad \cdots \end{array}. \tag{5.3}$$

According to our investigation Theorem 1.1 can be represented in the form.

Theorem 5.2. *Solution of the Neumann boundary value problem* (1.1) *for* $\varphi_j \in C^{k-j}(\partial S)$, $j = \overline{1,k}$ *exists then and only then when the vector* $b_k = (b_k^1, \ldots, b_k^k)$ *which is equal to the* k*th row of the triangle* B *from* (5.3) *is orthogonal to the vector* $F_k = \left(\int_{\partial S}\varphi_1(s)\,ds, \ldots, \int_{\partial S}\varphi_k(s)\,ds\right)$, *i.e.*

$$(b_k, F_k) = \int_{\partial S}\sum_{j=1}^{k}(-1)^{j+1}\binom{2k-j-1}{j-1}(2k-2j-1)!!\,\varphi_j(s)\,ds = 0. \tag{5.4}$$

For example, the solvability condition for the Neumann boundary value problem for the 5-harmonic equation in the unit ball has the form

$$\int_{\partial S}\left(105\varphi_1(s) - 105\varphi_2(s) + 45\varphi_3(s) - 10\varphi_4(s) + \varphi_5(s)\right)ds = 0.$$

Constructing solutions of the Neumann boundary value problem (1.1) for the polyharmonic equation can be reduced to the Dirichlet boundary value problem [(4),(7),(8)].

6 CONCLUSIONS

The major findings of the study are

a New arithmetical triangle B given by (5.3) is introduced and investigated.

b Solution of the recurrence relation $b_{k+1}^s = (2k-s)b_k^s - b_k^{s-1}$ under the boundary conditions $b_k^1 = \dfrac{(2k-1)!!}{(2k-1)}$ and $b_k^k = (-1)^{k+1}$, $k \in \mathbb{N}$ is obtained.

c Solvability condition of the Neumann boundary value problem (1.1) for the polyharmonic equation in the unit ball is explicitly given in the form (5.4) .

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Competing Interests

The authors declare that no competing interests exist.

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