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Article

On Backlund transformation of Riccati equation method and its application to nonlinear partial differential equations and differential-difference equations

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Abstract: In this paper, we investigate the equivalence between the Backlund transformation of Riccati equation method and the extended tanh-function method. It is proved that the two methods are equivalent when applying them to partial differential equations and differential-difference equations. Two examples are introduced to justify our results.

Keywords: Extended tanh-function method, Backlund transformation of Riccati equation method, partial differential equations, differential-difference equations.

MSC: 35C07, 35Q51.

1. Introduction

Many physical, biological and chemical phenomena can be modeled using partial differential equations (PDEs) and differential-difference equations (DDEs). So, in the last decades, many researchers have been interested in obtaining exact solutions of PDEs and DDEs. Many methods were proposed for achieving this task. Some of these methods are: the tanh method [1], the extended tanh-function method (ETM) [2], the simplest equation method [3], the integral bifurcation method [4], the extended mapping transformation method [5,6] and the Backlund transformation of Riccati equation method (BTREM) [7–12]. Our objective in this paper is to investigate the equivalence between the BTREM and the ETM.

2. Description of the two methods

In the following two subsections we give a brief description of the two methods.

2.1. The extended tanh-function method [2]

Consider a given partial differential equation with some independent variables, say, x and t and dependent variable u:

$$H(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, u_x and u_t are the derivatives of u with respect to x and t respectively. It is assumed that the Equation (1) has the following traveling wave solution:

$$u = u(z), \quad z = kx + ct + z_0, \tag{2}$$

where k, c and z_0 are some constants. Substituting Equation (2) into Equation (1), we get the following reduced ordinary differential equation:

$$H(u, u', u'', ...) = 0, \tag{3}$$

where the primes denote the derivative with respect to z. The solution of the Equation (3) can be expressed as:

$$u = \sum_{i=0}^{n} a_i \phi^i(z), \tag{4}$$

where $a_i, i = 0, 1, 2, , n$ are some constants that will be computed later, n is a positive integer computed by the balance between the highest-order derivative term and the nonlinear terms in the Equation (3) and ϕ satisfies the following Riccati equation:

$$\phi'(z) = \sigma + \phi^2(z), \tag{5}$$

where σ is a constant. The Riccati Equation (5) has the following solutions:

1. If $\sigma < 0$

$$\phi(z) = -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}z), \tag{6}$$

$$\phi(z) = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}z), \tag{7}$$

2. If $\sigma = 0$

$$\phi(z) = -\frac{1}{z+\omega}, \quad \omega = const. \tag{8}$$

3. If $\sigma > 0$

$$\phi(z) = \sqrt{\sigma}\tan(\sqrt{\sigma}z), \tag{9}$$

$$\phi(z) = -\sqrt{\sigma}\cot(\sqrt{\sigma}z). \tag{10}$$

Substituting Equation (4) into Equation (3) and making use of Equation (5), then setting the coefficients of $\phi^i(z), i = 0, 1, ...$ to zero, we get a set of algebraic equations for $a_i, i = 0, 1, 2, ..., n$. Solving this obtained system will lead to the values of $a_i, i = 0, 1, 2, ..., n$.

2.2. Backlund transformation of Riccati equation method [10]

In this method the solution of the Equation (3) is given in the form:

$$u = \sum_{i=0}^{n} b_i \Phi^i(z), \tag{11}$$

where $\Phi(z)$ is given by:

$$\Phi(z) = \frac{-\sigma B + D\phi(z)}{D + B\phi(z)}, \tag{12}$$

$b_i, i = 0, 1, 2, , n$ are some constants that will be computed later, n is a positive integer computed by the balance between the highest-order derivative term and nonlinear terms in the Equation (3), $\phi(z)$ are the known solutions of Ricatti Equation (5), B and D are arbitrary constants. Substituting Equation (11) into Equation (3), then setting the coefficients of $\phi(z)$ to zero, we get some algebraic equations for $b_i, i = 0, 1, 2, ..., n$. Solving this system of algebraic equations will lead to the values of $b_i, i = 0, 1, 2, ..., n$.

3. Equivalence of the two methods

Case 1: when $\phi(z) = -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}z)$. In this case, we have

$$\Phi(z) = \frac{-\sigma B - D\sqrt{-\sigma}\tanh(\sqrt{-\sigma}z)}{D - B\sqrt{-\sigma}\tanh(\sqrt{-\sigma}z)} = \sqrt{-\sigma}\frac{\frac{B\sqrt{-\sigma}}{D} - \tanh(\sqrt{-\sigma}z)}{1 - \frac{B\sqrt{-\sigma}}{D}\tanh(\sqrt{-\sigma}z)}. \tag{13}$$

By assuming that $\left(-\frac{B\sqrt{-\sigma}}{D}\right) = \tanh(k_1)$, k_1 is a constant, we get $k_1 = \tanh^{-1}\left(-\frac{B\sqrt{-\sigma}}{D}\right)$. Therefore,

$$\Phi(z) = -\sqrt{-\sigma}\frac{\tanh(\sqrt{-\sigma}z) + \tanh(k_1)}{1 + \tanh(k_1)\tanh(\sqrt{-\sigma}z)} = -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}z + k_1). \tag{14}$$

It is clear that $\Phi(z)$ and $\phi(z)$ are only differed by the constant phase shift k_1.

Case 2: when $\phi(z) = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}z)$. In this case, we get

$$\Phi(z) = \frac{-\sigma B - D\sqrt{-\sigma}\coth(\sqrt{-\sigma}z)}{D - B\sqrt{-\sigma}\coth(\sqrt{-\sigma}z)} = \sqrt{-\sigma}\frac{1 - \frac{D}{B\sqrt{-\sigma}}\coth(\sqrt{-\sigma}z)}{\frac{D}{B\sqrt{-\sigma}} - \coth(\sqrt{-\sigma}z)}. \tag{15}$$

By setting $\left(-\frac{D}{B\sqrt{-\sigma}}\right) = \coth(k_2)$, k_2 is a constant, we obtain $k_2 = \coth^{-1}\left(-\frac{D}{B\sqrt{-\sigma}}\right)$. Therefore,

$$\Phi(z) = -\sqrt{-\sigma}\frac{1+\coth(k_2)\coth(\sqrt{-\sigma}z)}{\coth(\sqrt{-\sigma}z)+\coth(k_2)} = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}z+k_2). \quad (16)$$

It is clear that $\Phi(z)$ and $\phi(z)$ are only differed by the constant phase shift k_2.
Case 3: when $\phi(z) = -\frac{1}{z+\omega}$, $\sigma = 0$. In this case, we have

$$\Phi(z) = \frac{D\left(-\frac{1}{z+\omega}\right)}{D+B\left(-\frac{1}{z+\omega}\right)} = \frac{-D}{Dz+D\omega-B} = \frac{-1}{z+\omega-\frac{B}{D}}. \quad (17)$$

It is clear that $\Phi(z)$ and $\phi(z)$ are only differed by the constant $-\frac{B}{D}$.
Case 4: when $\phi(z) = \sqrt{\sigma}\tan(\sqrt{\sigma}z)$. In this case, we get

$$\Phi(z) = \frac{-\sigma B + D\sqrt{\sigma}\tan(\sqrt{\sigma}z)}{D+B\sqrt{\sigma}\tan(\sqrt{\sigma}z)} = \sqrt{\sigma}\frac{-\frac{B\sqrt{\sigma}}{D}+\tan(\sqrt{\sigma}z)}{1+\frac{B\sqrt{\sigma}}{D}\tan(\sqrt{\sigma}z)}. \quad (18)$$

Assuming that $\left(-\frac{B\sqrt{\sigma}}{D}\right) = \tan(k_3)$, k_3 is a constant, we get $k_3 = \tan^{-1}\left(-\frac{B\sqrt{\sigma}}{D}\right)$. Therefore,

$$\Phi(z) = \sqrt{\sigma}\frac{\tan(\sqrt{\sigma}z)+\tan(k_3)}{1-\tan(k_3)\tan(\sqrt{\sigma}z)} = \sqrt{\sigma}\tan(\sqrt{\sigma}z+k_3). \quad (19)$$

It is clear that $\Phi(z)$ and $\phi(z)$ are only differed by the constant phase shift k_3.
Case 5: when $\phi(z) = -\sqrt{\sigma}\cot(\sqrt{\sigma}z)$. In this case, we get

$$\Phi(z) = \frac{-\sigma B - D\sqrt{\sigma}\cot(\sqrt{\sigma}z)}{D-B\sqrt{\sigma}\cot(\sqrt{\sigma}z)} = -\sqrt{\sigma}\frac{1+\frac{D}{B\sqrt{\sigma}}\cot(\sqrt{\sigma}z)}{\frac{D}{B\sqrt{\sigma}}-\cot(\sqrt{\sigma}z)}. \quad (20)$$

By setting $\left(-\frac{D}{B\sqrt{\sigma}}\right) = \cot(k_4)$, k_4 is a constant, we get $k_4 = \cot^{-1}\left(-\frac{D}{B\sqrt{\sigma}}\right)$. Therefore,

$$\Phi(z) = -\sqrt{\sigma}\frac{\cot(k_4)\cot(\sqrt{\sigma}z)-1}{\cot(\sqrt{\sigma}z)+\cot(k_4)} = -\sqrt{\sigma}\cot(\sqrt{\sigma}z+k_4). \quad (21)$$

It is clear that $\Phi(z)$ and $\phi(z)$ are only differed by the constant phase shift k_4.

4. The Drinfeld-Sokolov-Wilson equation

The Drinfeld-Sokolov-Wilson equation is given by [10]

$$u_t + Pvv_x = 0, \quad v_t + ruv_x + su_xv + qv_{xxx=0}, \quad (22)$$

where p, q, r and s are some nonzero constants. The authors in [10] have introduced the traveling wave transformation:

$$u(x,t) = U(z), \quad v(x,t) = V(z), \quad z = k(x-ct), \quad (23)$$

where k and c are constants. Substituting Equation (23) into Equation (22), we obtain the following ordinary differential equations:

$$-kcU' + pkVV' = 0, \quad -kcV' + rkUV' + skU'V + qk^3V''' = 0. \quad (24)$$

After applying the BTREM in [10], the authors have obtained four solutions for the Equation (22). These four solutions are the same solutions obtained in [13] as will be shown in the following discussion.
The first solution is given by:

$$
\begin{aligned}
u_1(x,t) &= \frac{6c}{r+2s}\left(\frac{\sqrt{\frac{-c}{2qk^2}}B - D\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{D-\sqrt{\frac{-c}{2qk^2}}B\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right)^2 \\
&= \frac{6c}{r+2s}\left(\frac{-\sqrt{\frac{-c}{2qk^2}}\frac{B}{D}+\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{1-\sqrt{\frac{-c}{2qk^2}}\frac{B}{D}\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right)^2. \qquad (25)
\end{aligned}
$$

Assume that $\left(-\sqrt{\frac{-c}{2qk^2}}\frac{B}{D}\right) = \tanh(k_5)$, k_5 is a constant. Therefore,

$$
\begin{aligned}
u_1(x,t) &= \frac{6c}{r+2s}\left(\frac{\tanh(k_5)+\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{1+\tanh(k_5)\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right)^2 \\
&= \frac{6c}{r+2s}\left(\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct)+k_5)\right)\right)^2, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
v_1(x,t) &= \pm\sqrt{\frac{12c^2}{p(r+2s)}}\left(\frac{\sqrt{\frac{-c}{2qk^2}}B - D\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{D-\sqrt{\frac{-c}{2qk^2}}B\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right) \\
&= \pm\sqrt{\frac{12c^2}{p(r+2s)}}\left(\tanh\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct)+k_5)\right)\right), \qquad (27)
\end{aligned}
$$

which is the same solution given in [13]. They are only differed by the phase shift constant k_5.

The second solution is given by:

$$
\begin{aligned}
u_2(x,t) &= \frac{6c}{r+2s}\left(\frac{\sqrt{\frac{-c}{2qk^2}}B - D\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{D-\sqrt{\frac{-c}{2qk^2}}B\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right)^2 \\
&= \frac{6c}{r+2s}\left(\frac{1-\sqrt{\frac{-2qk^2}{c}}\frac{D}{B}\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{-\sqrt{\frac{-2qk^2}{c}}\frac{D}{B}+\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right)^2. \qquad (28)
\end{aligned}
$$

Let $\left(-\sqrt{\frac{-2qk^2}{c}}\frac{D}{B}\right) = \coth(k_6)$, k_6 is a constant. Therefore,

$$
\begin{aligned}
u_2(x,t) &= \frac{6c}{r+2s}\left(\frac{1+\coth(k_6)\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{\coth(k_6)+\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right)^2 \\
&= \frac{6c}{r+2s}\left(\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct)+k_6)\right)\right)^2, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
v_2(x,t) &= \pm\sqrt{\frac{12c^2}{p(r+2s)}}\left(\frac{\sqrt{\frac{-c}{2qk^2}}B - D\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}{D-\sqrt{\frac{-c}{2qk^2}}B\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct))\right)}\right)\\
&= \pm\sqrt{\frac{12c^2}{p(r+2s)}}\left(\coth\left(\sqrt{\frac{-c}{2qk^2}}(k(x-ct)+k_6)\right)\right), \qquad (30)
\end{aligned}
$$

which is the same solution given in [13]. They are only differed by the phase shift constant k_6.
The third solution is given by:

$$
\begin{aligned}
u_3(x,t) &= \frac{-6c}{r+2s}\left(\frac{-\sqrt{\frac{c}{2qk^2}}B + D\tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{D+\sqrt{\frac{c}{2qk^2}}B\tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right)^2\\
&= \frac{-6c}{r+2s}\left(\frac{-\sqrt{\frac{c}{2qk^2}}\frac{B}{D} + \tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{1+\sqrt{\frac{c}{2qk^2}}\frac{B}{D}\tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right)^2. \qquad (31)
\end{aligned}
$$

Assume that $\left(-\sqrt{\frac{c}{2qk^2}}\frac{B}{D}\right) = \tan(k_7)$, k_7 is a constant. Therefore,

$$
\begin{aligned}
u_3(x,t) &= \frac{-6c}{r+2s}\left(\frac{\tan(k_7) + \tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{1-\tan(k_7)\tanh\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right)^2\\
&= \frac{-6c}{r+2s}\left(\tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct)+k_7)\right)\right)^2, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
v_3(x,t) &= \pm\sqrt{\frac{-12c^2}{p(r+2s)}}\left(\frac{-\sqrt{\frac{c}{2qk^2}}B + D\tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{D+\sqrt{\frac{c}{2qk^2}}B\tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right)\\
&= \pm\sqrt{\frac{-12c^2}{p(r+2s)}}\left(\tan\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct)+k_7)\right)\right), \qquad (33)
\end{aligned}
$$

which is the same solution given in [13]. They are only differed by the phase shift constant k_7.
The forth solution is given by:

$$
\begin{aligned}
u_4(x,t) &= \frac{-6c}{r+2s}\left(\frac{\sqrt{\frac{c}{2qk^2}}B + D\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{-D+\sqrt{\frac{c}{2qk^2}}B\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right)^2\\
&= \frac{-6c}{r+2s}\left(\frac{-1-\sqrt{\frac{2qk^2}{c}}\frac{D}{B}\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{-\sqrt{\frac{2qk^2}{c}}\frac{D}{B}+\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right)^2. \qquad (34)
\end{aligned}
$$

Let $\left(-\sqrt{\frac{2qk^2}{c}}\frac{D}{B}\right) = \cot(k_8)$, k_8 is a constant. Therefore,

$$
u_4(x,t) = \frac{-6c}{r+2s}\left(\frac{-1+\cot(k_8)\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{\cot(k_8)+\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right)^2
$$

$$= \frac{-6c}{r+2s}\left(\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct)+k_8)\right)\right)^2, \tag{35}$$

$$\begin{aligned} v_4(x,t) &= \pm\sqrt{\frac{-12c^2}{p(r+2s)}}\left(\frac{\sqrt{\frac{c}{2qk^2}}B + D\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}{-D+\sqrt{\frac{c}{2qk^2}}B\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct))\right)}\right) \\ &= \pm\sqrt{\frac{-12c^2}{p(r+2s)}}\left(\cot\left(\sqrt{\frac{c}{2qk^2}}(k(x-ct)+k_8)\right)\right), \end{aligned} \tag{36}$$

which is the same solution given in [13]. They are only differed by the phase shift constant k_8.

5. The equivalence between the two methods when solving differential-difference equations

In this section, we also prove that the BTREM is equivalent to the ETM when applied to differential-difference equations. To achieve this task we choose the following example.

5.1. The discrete mKdV equation

The discrete mKdV equation is given by [14]:

$$\frac{\partial u_n(t)}{\partial t} = (\theta - u_n^2)(u_{n+1} - u_{n-1}), \tag{37}$$

where θ is a constant. To get the traveling wave solutions for Equation (37), the following transformation was introduced [14]:

$$u_n(t) = u(\xi_n), \quad \xi_n = dn + c_1 t + c_0, \tag{38}$$

to transform Equation (37) into:

$$c_1 u'(\xi_n) = (\theta - u_n^2(\xi_n))(u_{n+1}(\xi_n) - u_{n-1}(\xi_n)) \tag{39}$$

where d, c_1 and c_0 are constants. After using the BTREM, the authors in [14] have obtained the following solutions:

The first solution is given by:

$$u_1(\xi_n) = a_0 + a_1\frac{-rb + a\sqrt{r}\tan\left(\sqrt{r}\xi_n\right)}{a + b\sqrt{r}\tan\left(\sqrt{r}\xi_n\right)} = a_0 + a_1\sqrt{r}\frac{\frac{-rb}{a\sqrt{r}} + \tan\left(\sqrt{r}\xi_n\right)}{1 + \frac{b\sqrt{r}}{a}\tan\left(\sqrt{r}\xi_n\right)}, \tag{40}$$

where a, b, r, a_0 and a_1 are constants. Let $\left(\frac{-rb}{a\sqrt{r}}\right) = \tan(m_1)$, m_1 is a constant. Therefore,

$$u_1(\xi_n) = a_0 + a_1\sqrt{r}\frac{\tan(m_1) + \tan\left(\sqrt{r}\xi_n\right)}{1 - \tan(m_1)\tan\left(\sqrt{r}\xi_n\right)} = a_0 + a_1\sqrt{r}\tan\left(\sqrt{r}\xi_n + m_1\right), \tag{41}$$

which is a solution in the form of the tan function only with a constant phase shift m_1.

The second solution is given by:

$$u_2(\xi_n) = a_0 + a_1\frac{-rb - a\sqrt{r}\cot\left(\sqrt{r}\xi_n\right)}{a - b\sqrt{r}\cot\left(\sqrt{r}\xi_n\right)} = a_0 - a_1\sqrt{r}\frac{\frac{-a}{b\sqrt{r}}\cot\left(\sqrt{r}\xi_n\right) - 1}{\cot\left(\sqrt{r}\xi_n\right) - \frac{a}{b\sqrt{r}}}. \tag{42}$$

Let $\left(\frac{-a}{b\sqrt{r}}\right) = \cot(m_2)$, m_2 is a constant. Therefore,

$$u_2(\xi_n) = a_0 - a_1\sqrt{r}\frac{\cot(m_2)\cot\left(\sqrt{r}\xi_n\right) - 1}{\cot\left(\sqrt{r}\xi_n\right) + \cot(m_2)} = a_0 - a_1\sqrt{r}\cot\left(\sqrt{r}\xi_n + m_2\right), \tag{43}$$

which is a solution in the form of the cot function only with a constant phase shift m_2.

The third solution is given by:

$$u_3(\xi_n) = a_0 + a_1 \frac{-rb - a\sqrt{-r}\coth\left(\sqrt{-r}\xi_n\right)}{a - b\sqrt{-r}\coth\left(\sqrt{-r}\xi_n\right)} = a_0 - a_1\sqrt{-r}\,\frac{\frac{-a}{b\sqrt{-r}}\coth\left(\sqrt{-r}\xi_n\right) + 1}{\coth\left(\sqrt{r}\xi_n\right) - \frac{a}{b\sqrt{-r}}}. \tag{44}$$

Let $\left(\frac{-a}{b\sqrt{-r}}\right) = \coth(m_3)$, m_3 is a constant. Therefore,

$$u_3(\xi_n) = a_0 - a_1\sqrt{-r}\,\frac{\coth(m_3)\coth\left(\sqrt{-r}\xi_n\right) + 1}{\coth\left(\sqrt{-r}\xi_n\right) + \coth(m_3)} = a_0 - a_1\sqrt{-r}\coth\left(\sqrt{-r}\xi_n + m_3\right), \tag{45}$$

which is a solution in the form of the coth function only with a constant phase shift m_3.

6. Conclusion

We have proved that the the BTREM is equivalent to the ETM. We demonstrated this fact using two examples from partial differential equations and differential-difference equations.

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References

[1] Bekir, A. (2008). Applications of the extended tanh method for coupled nonlinear evolution equations. *Communications in Nonlinear Science and Numerical Simulation, 13*, 1748-1757.

[2] Fan, E. (2000). Extended tanh-function method and its applications to nonlinear equations. *Physics Letters A, 277*, 212-218.

[3] Ebaid, A., & Abd Elazem, N. Y. (2011). On the exact solutions of a nano boundary layer problem using the simplest equation method. *Physica Scripta, 84*, 065005.

[4] Bin, H., Weiguo, R., Can, C., & Shaolin, L. (2008). Exact travelling wave solutions of a generalized Camassa-Holm equation using the integral bifurcation method. *Applied Mathematics and Computation, 206*, 141-149.

[5] Zhao, H., & Cheng, L. B. (2005). Extended mapping transformation method and its applications to nonlinear partial differential equation(s). *Communications in Theoretical Physics, 44*, 473-478.

[6] Abdel Latif, M. S. (2011). Some exact solutions of KdV equation with variable coefiňĄcients. *Communications in Nonlinear Science and Numerical Simulation, 16*, 1783-1786.

[7] Hon, Y. C., Zhang, Y., & Mei, J. (2010). Exact solutions for differential-difference equations by backlund transformation of riccati equation. *Modern Physics Letters B, 24*,(27), 2713-2724.

[8] Arnous, A. H., Mirzazadeh, M., & Eslami, M. (2014). The Backlund transformation method of Riccati equation applied to coupled Higgs field and Hamiltonian amplitude equations. *Computational Methods for Differential Equations, 2* (4), 216-226.

[9] Arnous, A. H., Mirzazadeh, M., Moshokoa, S., Medhekar, S., Zhou, Q., Mahmood, M. F., Biswas, A. & Belic, M. (2015). Solitons in optical metamaterials with trial solution approach and Backlund transform of Riccati equation. *Journal of Computational and Theoretical Nanoscience, 12*(12), 5940-5948.

[10] Arnous, A. H., Mirzazadeh, M., & Eslami, M. (2016). Exact solutions of the Drinfel'd-Sokolov-Wilson equation using Backlund transformation of Riccati equation and trial function approach. *Pramana, 86*, 1153-1160.

[11] El-Borai, M. M., El-Owaidy, H. M., Ahmed, H. M., Arnous, A. H., & Mirzazadeh, M. (2017). Solitons and other solutions to the coupled nonlinear Schrodinger type equations. *Nonlinear Engineering, 6*(2), 115-121.

[12] Zayed, E. M., Alurrfi, K. A., & Al Nowehy, A. G. (2017). Many exact solutions of the nonlinear kpp equation using the backlund transformation of the Riccati equation. *International Journal of Optics and Photonic Engineering, 2*(1).

[13] Bibi, S., & Mohyud-Din, S. T. (2014). New traveling wave solutions of Drinefel'd- Sokolov- Wilson Equation using Tanh and Extended Tanh methods. *Journal of the Egyptian Mathematical Society, 22*, 517-523.

[14] Zhang, Y., Hon, Y. C., & Mei, J. (2010). A systematic method for solving differential-difference equations. *Communications in Nonlinear Science and Numerical Simulation, 15*, 2791-2797.